τ-Tilting Theory of Gentle Algebras

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Notations & Definitions:

- $k = \bar{k}$ a field.
- Q:= a finite quiver (directed graph).
- I:= an ideal of kQ, generated by paths of length more than 2.
- $A = kQ/I$:= a finite dim. alg.

String & Gentle Algebras: A is a ***string*** algebra, if

(i) for arrow α, at most $\exists!$ arrows β and γ s.t $\alpha\beta \notin I$ and $\gamma\alpha \notin I$.

(ii) at each vertex, $\exists$ at most two incoming and outgoing arrows.

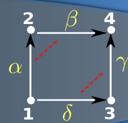
A is called ***gentle***, if additionally

(iii) A set of paths of length two in Q generates I.

(iv) for α, at most $\exists!$ arrows β and γ s.t $0 \neq \alpha\beta \in I$ and $0 \neq \gamma\alpha \in I$.

In a string algebra kQ/I, a ***string*** is (an orientation forgetting) walk in Q that neither goes through a relation, nor along an edge and then immidiately back along the same edge.

Example: $A = kQ/I$ is gentle:



τ-tilting Theory:

A-module X is called:

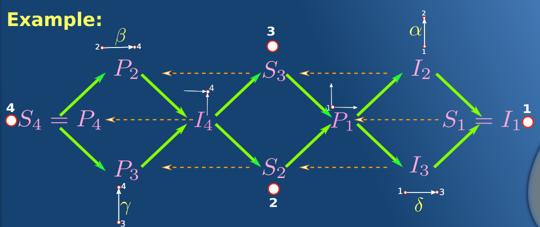
- τ-rigid if $Hom_A(X, \tau X) = 0$;
- τ-tilting if X is τ-rigid and $|X| = |A|$;
- Support τ-tilting if $\exists e = e^2 \in A$, s.t X is $A/\langle e\rangle$-tilting module.

Thm(BR): Let A be a string algebra.

(I) Indecomposables of $modA$ are string and band modules;

(II) Every indecomp. τ-rigid A-module is a string module.

Example:



$X = S_4 \oplus I_3 \oplus S_1$ is support τ-tilting, where $e = e_2$.

Notice, $X \longleftrightarrow \{ \overset{4}{\circ}, 1 \xrightarrow{\delta} 3, \overset{1}{\circ} \}$.

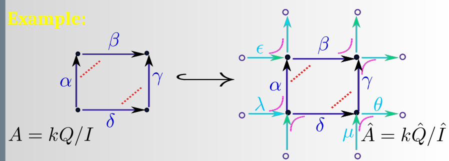
Fringed Algebras:

A vertex of a gentle bound quiver (Q, I) is ***defective*** if it does not have exactly two incoming and outgoing arrows.

Prop (BDMTY): Each gentle algebra $A = kQ/I$ canonically embeds into a gentle algebra $\hat{A} = k\hat{Q}/\hat{I}$ satisfying:

- Q is a subquiver of $\hat{Q}$ and $\hat{I}$ contains I;
- Every vertex of Q is non-defective in $\hat{Q}$.

Example:

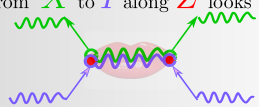


$$modA \hookrightarrow mod\hat{A}$$

$$Str(Q, I) \subset Str(\hat{Q}, \hat{I}).$$

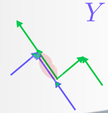
If X, Y are strings, a ***Kiss***

from X to Y along Z looks as:



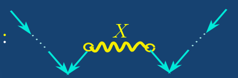
Lemma: Every kiss induces a morphism between the string modules!

Example: Let $X = \theta\delta\alpha^{-1}\epsilon$ and $Y = \mu^{-1}\delta\lambda$ then $Z = \delta$.



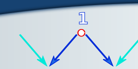
$cohook(X)$:= ***cohook completion*** of $X \in Str(A)$, in $(\hat{Q}, \hat{I})$ is obtained by adding the outgoing arrow to both ends of X and then longest incoming strings.

$cohook(X)$:



Thm(BDMTY): If X, Y in $Str(A)$:

- $\tau_{\hat{A}} Y = cohook(Y)$;
- $\tau_A Y$ is a submodule of $\tau_{\hat{A}} Y$.
- $Hom_{\hat{A}}(X, \tau_{\hat{A}} Y) \simeq Hom_A(X, \tau_A Y)$.



Example: $\tau_{\hat{A}}(S_1) = cohook(e_1) = \mu^{-1}\delta \quad \alpha^{-1}\epsilon$

$\tau_A(S_1) = P_1 = \delta\alpha^{-1}$

Thm(BDMTY): If A gentle and $X, Y \in Str(A)$,

- Elements of $Hom_A(X, \tau_A Y)$ and $Hom_{\hat{A}}(X, \tau_{\hat{A}} Y)$ corresponding to kisses from $cohook(X)$ to $cohook(Y)$ forms a basis.
- $\#kiss(cohook(X), cohook(Y)) = dimHom_{\hat{A}}(X, \tau_{\hat{A}} Y) = dimHom_A(X, \tau_A Y)$.
- $\exists$ a bijection

$$\left\{\begin{array}{c}\text{Basic Support}\\ \tau\text{- tilting}\\ \text{modules}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Maximal Collections}\\ \text{of}\\ \text{Non-Kissing Strings}\end{array}\right\}$$

References & Sponsors:

[BDMTY] T. Brüstle, G. Douville, K. Mousavand, H. Thomas, E. Yıldırım, *On combinatorics of gentle algebras*, arXiv:1707.07665.

[BR] M. C. R. Butler, C. M. Ringel, *Auslander-Reiten sequences with few middle terms*, Comm. in Algebra. 15 (1987).

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