DISTRIBUTION OF BRICKS AND AN OPEN ALGEBRO-GEOMETRIC CONJECTURE

KAVEH MOUSAVAND, CHARLES PAQUETTE

Abstract. In this note we are mainly concerned with the geometric behavior of bricks (Schur representations) over finite dimensional algebras. In particular, we first motivate and discuss two open conjectures which establish new connections between algebro-geometric aspects of representation theory and other areas, such as combinatorics, lattice theory and homological algebra. Then, we summarize some of the new results in this direction, including a theorem which significantly reduces our conjectures to a particular family of algebras. We also verify the conjectures for some important classes of algebras and outline our methodology to treat the general case. The last section comprises of some remarks on our ongoing research and future steps, as well as some related problems that could be investigated.

Contents

1. Motivations and Open conjectures

Below, k is an algebraically closed field, and Λ stands for a finite dimensional associative k-algebra with multiplicative identity 1_Λ. With no loss of generality we always assume Λ is basic and connected. Thus, the length of each Λ-module is the same as its dimension as a k-vector space. By $\operatorname{Mod}\Lambda$ we denote the category of all left Λ-modules, and $\operatorname{mod}\Lambda$ specifies the full subcategory of $\operatorname{Mod}\Lambda$ consisting of all finitely generated Λ-modules. Moreover, let $\operatorname{Ind}(\Lambda)$ and $\operatorname{ind}(\Lambda)$ respectively denote the collection of isoclasses of indecomposable modules in $\operatorname{Mod}\Lambda$ and $\operatorname{mod}\Lambda$. Under these standard assumptions, Λ is isomorphic to a quotient algebra kQ/I, where Q is a finite quiver (i.e, an oriented graph) and I is an admissible ideal in the path algebra kQ. Consequently, every X in $\operatorname{Mod}\Lambda$ can be viewed as a representation of the bound quiver (Q, I) and there is an equivalence between the categories $\operatorname{Mod}\Lambda$ and $\operatorname{Rep}(Q, I)$, where the latter denotes the category of all representations of (Q, I). This equivalence also restricts to that of $\operatorname{mod}\Lambda$ and $\operatorname{rep}(Q, I)$, and allows us to interchangeably use the terms modules and representations of Λ. Here $\operatorname{rep}(Q, I)$

2020 *Mathematics Subject Classification.* 16D80,16G20,16G60,05E10.

The second-named author was supported by the National Sciences and Engineering Research Council of Canada, and by the Canadian Defence Academy Research Programme.

denotes the full subcategory of $\operatorname{Rep}(Q,I)$ which consists of only finite dimensional representations of (Q,I). This also justifies the term *representation-finite* (or rep-finite, for short), used for those Λ for which $\operatorname{ind}(\Lambda)$ is finite. For the rudiments of representation theory of finite dimensional algebras, we refer to [ASS, SS], where the reader can also find all the standard terminology and materials used below.

A Λ-module M is called a *brick* if $\operatorname{End}_\Lambda(M)$ is a division algebra. Bricks are also known as *Schur representations* of the bound quiver (Q,I) associated to Λ. By $\operatorname{Brick}(\Lambda)$ and $\operatorname{brick}(\Lambda)$ we respectively denote the collection of isoclasses of bricks in $\operatorname{Mod}\Lambda$ and $\operatorname{mod}\Lambda$. Evidently, $\operatorname{Brick}(\Lambda) \subseteq \operatorname{Ind}(\Lambda)$, and it is well-known that $\operatorname{Ind}(\Lambda) = \operatorname{Brick}(\Lambda)$ is equivalent to $\operatorname{ind}(\Lambda) = \operatorname{brick}(\Lambda)$, which implies Λ is rep-finite. We note if M is a brick, each nonzero endomorphism of M in $\operatorname{Mod}\Lambda$ is invertible. Because k is algebraically closed, M in $\operatorname{mod}\Lambda$ is a brick if and only if $\operatorname{End}_\Lambda(M) = k$. We say Λ is *brick-finite* provided $\operatorname{brick}(\Lambda)$ is finite. Thus, every rep-finite algebra is evidently brick-finite, but the converse is not true. In fact, each rep-infinite local algebra admits a unique brick (for example, consider $\Lambda = k[x,y]/J$, where J is the ideal generated by $\{x^2, xy, yx, y^2\}$.). Hence, for a family $\mathcal{F}$ of algebras, it is an interesting (and *a priori* hard) problem to find explicit criteria to decide which algebras in $\mathcal{F}$ are brick-finite and which ones are not. As discussed below, we view brick-finiteness as a natural generalization of representation finiteness. As briefly discussed below, there are strong combinatorial, algebro-geometric and homological motivations and arguments which support this perspective.

Since the problem of brick (in)finiteness is interesting only for representation infinite algebras, henceforth we implicitly assume Λ is representation infinite. In particular, for a family $\mathcal{F}$ of rep-infinite algebras, our primarily goal is to use combinatorial, homological, lattice theoretical and algebro-geometric realizations of brick (in)finiteness and enrich the dictionary between these domains. This diverse range of tools also allows us to articulate our open conjectures in different languages. In doing so, we mainly use [AIR], [DIJ], [DI+], and [CKW]. In particular, the more advanced techniques and conceptual interactions between different domains which are treated here rely on the aforementioned papers. Before we can state our conjectures and results, we need to fix some notations.

For each quiver Q, by Q_0 and Q_1 we respectively denote the set of vertices and arrows of Q. Observe that the rank of the Grothendieck group of Λ is $|Q_0|$. A *dimension vector* $\underline{d}$ of Λ is just a vector in the positive cone of the Grothendieck group. That is $\underline{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$, and the affine variety $\operatorname{mod}(\Lambda, \underline{d})$ consists of all representations of (Q,I) whose dimension vector is $\underline{d}$, on which the general linear group $\operatorname{GL}(\underline{d})$ acts via conjugation. If $\mathcal{O}_M$ denotes the $\operatorname{GL}(\underline{d})$-orbit of M in $\operatorname{mod}(\Lambda, \underline{d})$, each point in $\mathcal{O}_M$ corresponds to a Λ-module which is isomorphic to M. The collection of irreducible components of $\operatorname{mod}(\Lambda, \underline{d})$ is denoted by $\operatorname{Irr}(\Lambda, \underline{d})$, and $\mathcal{Z} \in \operatorname{Irr}(\Lambda, \underline{d})$ is a *brick component* if it contains a brick. These are of especial interest in our work.

Following [CKW], we say Λ has the *dense orbit property* if for every dimension vector $\underline{d}$, and each irreducible component $\mathcal{Z}$ in $\operatorname{mod}(\Lambda, \underline{d})$, there exists a dense orbit in $\mathcal{Z}$. Moreover, we say Λ is *brick discrete* if for each $\underline{d}$, there are only finitely many orbits of bricks in $\operatorname{mod}(\Lambda, \underline{d})$. We introduce this new terminology to avoid the confusion caused by the conflict of terminology. In particular, we remark that what we called "brick discrete" is named "Schur-representation-finite" in [CKW].

A Λ-module M is *rigid* if $\operatorname{Ext}^1_\Lambda(M,M) = 0$, and it is called *$\tau$-rigid* provided $\operatorname{Hom}_\Lambda(M, \tau_\Lambda M) = 0$. Here, τ_Λ denotes the Auslander-Reiten translation, and we

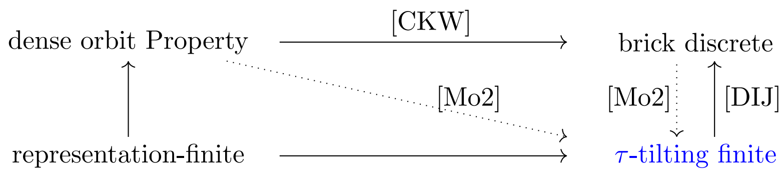


FIGURE 1. The map of our work, where the conjectural implications are depicted by dotted arrows.

often suppress Λ from the notation. By τ-rigid(Λ) we denote the set of isomorphism classes of basic τ-rigid modules. A rigid module X is called *tilting* if its projective dimension is at most one and $|X| = |\Lambda|$. Here, $|M|$ denotes the number of non-isomorphic indecomposable summands of M in $\operatorname{Mod}\Lambda$. Analogously, a τ-rigid module M is *τ-tilting* if $|M| = |\Lambda|$. By $\operatorname{tilt}(\Lambda)$ and τ-$\operatorname{tilt}(\Lambda)$ we respectively denote the set of all isomorphism classes of basic tilting modules, and that of all basic τ-tilting modules in $\operatorname{mod}\Lambda$. In [AIR], the author introduced a modern setup where many rich ideas from cluster algebras meet with the classical tilting theory. This also fixed the deficiency of the classical tilting theory with respect to the mutation of tilting modules (see [BB]). This goal was accomplished by conceptualizing the notion of mutation of clusters in terms of some homological properties of (support) τ-tilting modules. For details, see [AIR].

Now we can state the first version of our main conjecture.

Conjecture 1.1 ([Mo2]) *For each algebra Λ, the following hold:*

(1) If Λ has the dense orbit property, then it is τ-tilting finite.
(2) Λ is brick discrete if and only if Λ is τ-tilting finite.

This conjecture is depicted in Figure 1, which puts our investigations and results in perspective by relating them to the recent work in [CKW] and [DIJ]. Both parts of this conjecture first appeared in [Mo2], while the first-named author studied the τ-tilting finiteness of algebras in his doctoral dissertation.

Because τ-tilting modules are the main ingredient of the modern τ-tilting theory, finding nontrivial conditions such that $|\tau\text{-}\operatorname{tilt}(\Lambda)| < \infty$ is monumental and has spurred a lot of research in various areas, including homological algebra, combinatorics, lattice theory and geometry (for example, see [DIJ], [DI+], [Mo1], [Pl], [KPY], and the references therein). In Section 2, we mention some of the key results that are closely related to the scope of our work, including the "brick τ-rigid correspondence" from [DIJ], which implies $|\operatorname{brick}(\Lambda)| < \infty$ if and only if $|\tau\text{-}\operatorname{tilt}(\Lambda)| < \infty$. This allows us to rephrase Conjecture 1.1 in the combinatorial terms, as follows.

Conjecture 1.2 *For each algebra Λ, the following hold:*

(1) If Λ has the dense orbit property, then it is brick-finite.
(2) Λ is brick infinite if and only if there is an infinite family of non-isomorphic bricks of length d, for a positive integer d.

In [Mo1, Mo2], the first-named author settled the above conjectures for some cases and outlined a long-term project to treat the general case. In [MP], we prove an important reductive theorem to approach this goal. We discuss these results in Section 2. We also observe that the second part of this version of our main conjecture also appears in [ST] and is treated for special biserial algebras.

2. Methodology and Main results

Motivated by the conjectures 1.1 and 1.2, we are only interested in the behavior of representation infinite algebras, and particularly in the size of brick(Λ) and distributions of bricks over such algebras. Recall that Λ is *minimal representation-infinite* (or min-rep-inf, for short) provided Λ is representation infinite but for every nonzero ideal J of Λ, the quotient algebra Λ/J is rep-finite. In the past 60 years, min-rep-inf algebras have played a pivotal role in representation theory of associative algebras. This is primarily due to their decisive role in the study of Brauer-Thrall Conjectures, which are among the cornerstones of modern representation theory (For the statements of these classical conjectures and more details, see [ASS, IV.5] and [Bo2].). Thanks to the recent elegant classification of Ringel [Ri], together with a small observation in [Mo1], we know that every min-rep-inf algebra falls into (at least) one of the following three subfamilies:

- $\mathrm{Mri}(\mathfrak{F}_{\mathrm{B}})$: min-rep-inf **B**iserial algebras;
- $\mathrm{Mri}(\mathfrak{F}_{\mathrm{nD}})$: min-rep-inf **n**on-**D**istributive algebras;
- $\mathrm{Mri}(\mathfrak{F}_{\mathrm{gC}})$: min-rep-inf algebras with a **g**ood **C**overing $\widetilde{\Lambda}$ and a finite convex subcategory of $\widetilde{\Lambda}$ is tame-concealed of type $\widetilde{\mathbb{D}}_n$ or $\widetilde{\mathbb{E}}_{6,7,8}$.

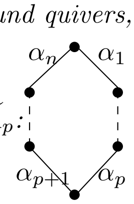
Recall that Λ is *biserial* if for any left or right non-uniserial indecomposable projective module P, the radical of P is a sum of two uniserial submodules X and Y such that $X \cap Y$ is either zero or a simple module. Since $\Lambda = kQ/I$ is basic, this is equivalent to saying that for each $x \in Q_0$ and the associated indecomposable (left or right) projective module P_x, we must have $\mathrm{rad}(P_x) = M + N$ with M and N uniserial and $\dim_k(M \cap N) \leq 1$. Moreover, Λ is said to be *distributive* if the lattice of two-sided ideals in Λ is distributive. In [Ja], the author shows Λ is distributive if and only if this lattice is finite. Moreover, all rep-finite algebras are known to be distributive. Those algebras in $\mathrm{Mri}(\mathfrak{F}_{\mathrm{gC}})$ are characterized in terms of the properties of their Galois covering, as studied in [BG+].

In 2013 and 2018, Ringel and Bongartz respectively described $\mathrm{Mri}(\mathfrak{F}_{\mathrm{B}})$ and $\mathrm{Mri}(\mathfrak{F}_{\mathrm{nD}})$ in terms of their bound quivers. In contrast, it is known that the algebras in $\mathrm{Mri}(\mathfrak{F}_{\mathrm{gC}})$ have no explicit description (for further details, see [Bo2, Bo3]). Based on the aforementioned elegant classifications in [Ri] and [Bo2], the first-named author studied the brick (in)finiteness of min-rep-inf algebras, and fully determined which algebras in $\mathrm{Mri}(\mathfrak{F}_{\mathrm{B}}) \cup \mathrm{Mri}(\mathfrak{F}_{\mathrm{nD}})$ are brick-finite and which ones are not. In the following quivers, those edges which are not directed can take any orientation, and dashed segments could be of any length and orientations of arrows.

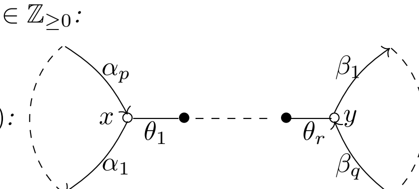
Theorem 2.1 ([Mo1, Mo2]) *Let $\Lambda = kQ/I$ be a minimal representation-infinite algebra.*

(1) If Λ is biserial, then it is brick infinite if and only if (Q, I) is one of the following bound quivers, where $n, p, q, r \in \mathbb{Z}_{\geq 0}$:

acyclic $\widetilde{A}_p$:



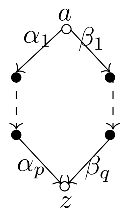
$B(p,q,r)$:



$I := \langle \alpha_1\alpha_p, \beta_1\beta_q \rangle$

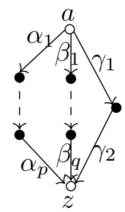
(2) If Λ is non-distributive, then Λ is brick infinite if and only if $f(Q,I)$ is one of the following bound quivers, with the specified $p,q \in \mathbb{Z}$:

A(p, q):



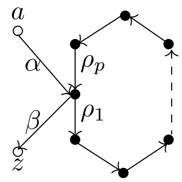
$p,q \in \mathbb{Z}_{>0}$
$I_A := 0$

B(p, q):



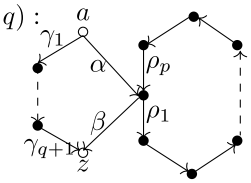
$2 = q \leq p$ *or* $3 = q \leq p \leq 5$
$I_B := \langle \gamma_2\gamma_1 + \beta_q \cdots \beta_1 + \alpha_p \cdots \alpha_1 \rangle$

$C(p)$:



$p \in \mathbb{Z}_{>0}$
$I_C := \langle \rho_1\rho_p \rangle$

$D(p,q)$:



$p,q \in \mathbb{Z}_{>0}$
$I_D := \langle \rho_1\rho_p, \gamma_{q+1} \cdots \gamma_1 - \beta\alpha \rangle$

In [Ri], those min-rep-inf algebras whose bound quiver is of the form $B(p,q,r)$ are called *barbell*. For each barbell algebra, the minimality assumption implies that the bar between vertices x and y is not serial, meaning that the orientation of all θ_i's cannot be in the same direction. Moreover, in a barbell algebra, if $\alpha_p \cdots \alpha_1$ and $\beta_q \cdots \beta_1$ are both cyclic paths in $B(p,q,r)$, we must have $r > 0$, as otherwise we get an infinite dimensional algebra.

Thanks to the preceding theorem, we verify Conjecture 1.1 for the entire family $\mathrm{Mri}(\mathfrak{F}_{\mathrm{B}}) \cup \mathrm{Mri}(\mathfrak{F}_{\mathrm{nD}})$. In fact, we prove the following stronger result.

Corollary 2.2 ([Mo2]) *Let Λ be a biserial or non-distributive minimal representation-infinite algebra. Then, Λ is brick infinite if and only if there exists a one-parameter family $\{M_\lambda\}_{\lambda \in k^*}$ of non-isomorphic bricks in* $\mathrm{mod}\,\Lambda$ *which are of the same length.*

As a byproduct of our methodology that lead to Theorem 2.1, we obtained that a gentle algebra is brick-finite if and only if it is rep-finite (for the terminology and further details, see [Mo1]). This result was also independently shown in [Pl]. In fact, in retrospect, we proved that a min-rep-inf biserial algebra is brick infinite if and only if it is gentle. And, for gentle algebras we can verify a stronger version of Conjecture 1.2. This, together with the recent results form [GL+], implies that Λ in $\mathrm{Mri}(\mathfrak{F}_{\mathrm{B}})$ is brick infinite if and only if there is a brick component $\mathcal{Z}$ in $\mathrm{Irr}(\Lambda)$ and a rational curve $\mathcal{C}$ of non-isomorphic bricks $\{M_\lambda\}$ in $\mathcal{Z}$ with $\mathcal{Z} = \overline{\bigcup_{\lambda \in \mathcal{C}} \mathcal{O}_{\mathcal{M}_\lambda}}$. Here, the closure is with respect to Zariski topology.

For an arbitrary algebra Λ, deciding whether or not the conjectures 1.1 and 1.2 hold seems to be currently out of reach. However, we can reduce our conjecture to *minimal brick infinite* algebras. Those are the algebras Λ such that Λ is brick infinite but every proper quotient algebra Λ/J is brick-finite. This is the modern analogue of the notion of minimal representation infinite algebra. The first reduction to minimal brick infinite algebras relies on a well-known fact: for any pair of algebras A and B, and each epimorphism of algebras $\psi : A \to B$, we get a full embedding of $\operatorname{mod} B$ into $\operatorname{mod} A$. In particular, $\operatorname{ind}(B) \subseteq \operatorname{ind}(A)$ and $\operatorname{brick}(B) \subseteq \operatorname{brick}(A)$ always hold. Before we state some of our new results on minimal brick infinite algebras and further reduce our conjecture, we recall some important notions and fundamental theorems which play crucial roles in our work.

A subcategory $\mathcal{T}$ of $\operatorname{mod}\Lambda$ is called a *torsion class* if it is closed under quotient and extension. A torsion class $\mathcal{T}$ is *functorially finite* if $\mathcal{T} = \operatorname{Fac}(M)$, for some Λ-module M, where $\operatorname{Fac}(M)$ denotes the full subcategory of $\operatorname{mod}\Lambda$ consisting of all modules which are factor modules of a finite direct sum of M. By $\operatorname{tors}(\Lambda)$ we denote the set of all torsion classes in $\operatorname{mod}\Lambda$, and $\operatorname{f-tors}(\Lambda)$ consists of all those $\mathcal{T}$ in $\operatorname{tors}(\Lambda)$ which are functorially finite. Observe that $\operatorname{tors}(\Lambda)$ comes with a natural lattice structure. We note that intersection of any family of torsion classes in $\operatorname{tors}(\Lambda)$ belongs to $\operatorname{tors}(\Lambda)$. Hence, for $\mathcal{T}_1$ and $\mathcal{T}_2$ in $\operatorname{tors}(\Lambda)$, we define the meet of $\mathcal{T}_1$ and $\mathcal{T}_2$ as $\mathcal{T}_1 \cap \mathcal{T}_2$, whereas their join is considered to be the intersection of all torsion classes in $\operatorname{tors}(\Lambda)$ that contain $\mathcal{T}_1 \cup \mathcal{T}_2$. Although $\operatorname{f-tors}(\Lambda)$ inherits this partial order from $\operatorname{tors}(A)$, in general $\operatorname{f-tors}(\Lambda)$ is not a lattice. The lattice theoretical properties of $\operatorname{tors}(\Lambda)$ and the profound applications of the poset $\operatorname{f-tors}(\Lambda)$ in τ-tilting theory has been extensively studied in [DI+]. Below, we only summarize some of the key results used in this note.

Theorem 2.3 [DIJ, DI+] *For an algebra Λ, the following are equivalent.*

(1) Λ *is τ-tilting finite;*
(2) $\operatorname{f-tors}(\Lambda) = \operatorname{tors}(\Lambda)$*;*
(3) $\operatorname{tors}(\Lambda)$ *is finite;*
(4) $\tau\text{-rigid}(\Lambda)$ *is finite;*
(5) $\operatorname{brick}(\Lambda)$ *is finite.*

From the preceding theorem, one immediately observes that every minimal brick infinite algebra is in fact *minimal τ-tilting infinite* (or min-τ-inf, for short). That is, Λ is τ-tilting infinite but every proper algebra quotient of Λ is τ-tilting finite. This, as well as other equivalences in Theorem 2.3, allowed us to employ various technologies from homological algebra, lattice theory and combinatorics, while studying Conjecture 1.1 in the general case. In other words, our ultimate goal is to give an algebro-geometric incarnation of τ-tilting finiteness and through that establish novel connections to the important realizations of this notion in other areas, as depicted in Figure 1.

In [MP], we delved into these connections and used the modern concept of min-τ-inf algebras both in the treatment of our open conjectures 1.1 and 1.2, as well as in the more classical setting of tilting theory. Before we state our main reduction theorem, let us give analogous characterizations of the classical and modern minimality conditions discussed in this note. To do so, we first recall that a module is *faithful* if its annihilator is trivial. As observed in [MP], from a known result of Auslander [Au] and the recent work of Sentieri [Se], we have

- Λ is minimal representation infinite if and only if $\operatorname{Ind}(\Lambda) \setminus \operatorname{ind}(\Lambda) \neq \emptyset$ and every $M \in \operatorname{Ind}(\Lambda) \setminus \operatorname{ind}(\Lambda)$ is faithful.
- Λ is minimal τ-tilting infinite if and only if $\operatorname{Brick}(\Lambda) \setminus \operatorname{brick}(\Lambda) \neq \emptyset$ and every $N \in \operatorname{Brick}(\Lambda) \setminus \operatorname{brick}(\Lambda)$ is faithful.

The next theorem lists some important properties of min-τ-infinite algebras. As explained in [MP], this theorem also highlights some fundamental differences and similarities between the min-τ-infinite algebras and the min-rep-inf algebras. We recall that a vertex v in (Q, I) is said to be a *node* if it is neither a sink nor a source, and for each pair of arrows α and β in Q with $e(\alpha) = v = s(\beta)$, we have $\beta\alpha \in I$. Then, $\Lambda = kQ/I$ is called *node-free* if (Q, I) has no nodes. Moreover, Λ is called *central* if the center of Λ is k.

Theorem 2.4 *Let $\Lambda = kQ/I$ be a minimal τ-tilting infinite algebra. Then Λ is node-free and central. Moreover, Λ admits no projective-injective module.*

From the previous theorem, in [MP] we obtain an explicit characterization of τ-tilting finiteness of those algebras $\Lambda = kQ/I$ which are radical square zero. That is $\beta\alpha \in I$, for every pair of arrows α and β in Q. In particular, we easily recovered the following result of Adachi [Ad] and verified our conjectures 1.1 and 1.2 for this family of algebras. Below, we say Q is a *sink-source* quiver if every vertex of Q is either a sink or a source.

Corollary 2.5 ([Ad] and [MP]) *Let $\Lambda = kQ/I$ be such that* $\operatorname{rad}^2(\Lambda) = 0$. *The following are equivalent:*

(1) Λ is τ-tilting infinite;
(2) Q contains a sink-source subquiver of affine type.
(3) there exists a one-parameter family $\{M_\lambda\}_{\lambda \in k^*}$ of non-isomorphic bricks in $\operatorname{mod}\Lambda$ *which are of the same length.*

To highlight the significance of min-τ infinite algebras in the study of classical tilting theory, we first remark that unlike the representation finiteness and τ-tilting finiteness, the notion of tilting finiteness is not preserved under taking algebraic quotients. More precisely, there exist algebras Λ such that $\operatorname{mod}\Lambda$ contains only finitely many isomorphism classes of tilting modules, but for some ideal J in Λ, the quotient algebra Λ/J admits infinitely many ismorphism classes of tilting modules. Nevertheless, one can still define "minimal tilting infinite" algebras, analogous to the notions of min-rep-inf and min-τ-infinite algebras. Then, it is a natural question to ask whether this new family manifests any properties similar to min-rep-inf and min-τ-infinite algebras. We give an answer to this question in the next theorem. Let us recall that for a collection of objects $\mathfrak{O}$, we say *almost all* objects of $\mathfrak{O}$ satisfy property $\mathcal{P}$ provided all but finitely many objects of $\mathfrak{O}$ have property $\mathcal{P}$.

Theorem 2.6 *Let $\Lambda = kQ/I$ be an algebra. If Λ is minimal τ-tilting infinite, the projective dimension of almost all τ-rigid Λ-modules is exactly one. Thus,*

(1) Λ is minimal τ-tilting infinite if and only if it is minimal tilting infinite.
(2) If Λ is minimal tilting infinite, the mutation graph of tilting modules in $\operatorname{mod}\Lambda$ *is of degree $|Q_0|$ at almost every vertex.*

This additional knowledge of min-τ-infinite algebras, and some further geometric techniques we develop in [MP], allowed us to obtain a better understanding of orbits of bricks and brick components of brick infinite algebras. This also resulted in a conceptual proof for an unboundedness theorem on the length of bricks over brick infinite algebras: If Λ is brick infinite, for each $d \in \mathbb{Z}$ there is a brick M whose length is greater than d. This assertion was formerly shown in [STV] through different techniques. Our approach was based on the geometric study of "minimal extending brick" in $\operatorname{mod}\Lambda$, which are studied in [BCZ] and [DI+], particularly in the labelling of the edges of the lattice $\operatorname{tors}(\Lambda)$. More precisely, we showed that if M is a minimal extending brick of a functorially finite torsion class, then $\mathcal{O}_M$ is an open orbit, hence $\overline{\mathcal{O}}_M$ is a brick component in $\mathrm{Irr}(\Lambda)$. For details on minimal extending bricks and our geometric approach to their study, see respectively [DI+] and [MP].

We finish this section by the following reduction theorem in the study of Conjecture 1.1. Before we state that, note that a min-τ-infinite algebra may admit infinitely many non-isomorphic bricks with distinct annihilators. However, we prove that it is sufficient to only treat Conjectures 1.1 and 1.2 for a particular subfamily of brick infinite algebras, as stated in the following theorem.

Theorem 2.7 *Conjecture 1.1 (and therefore Conjecture 1.2) holds in general if and only if it is true for each minimal τ-tilting infinite algebras for which almost all bricks are faithful. In particular, if a minimal τ-tilting infinite algebra has infinitely many unfaithful bricks, then it admits an infinite family of bricks of the same length.*

3. Future steps and some related problems

As mentioned in Section 1, minimal representation infinite algebras played a crucial role in the treatment of the celebrated Brauer-Thrall Conjectures. They were vastly studied to prove that any rep-infinite algebras is both unbounded and strongly unbounded. That is to say, if Λ is rep-infinite, there is no bound on the length of modules in $\mathrm{ind}(\Lambda)$, whereas strongly unboundedness asserts there exists an infinite sequence of positive integers $d_1 < d_2 < d_3 < \cdots$ such that for each d_i there are infinitely many (isomorphism classes of) indecomposable Λ-modules of length d_i. In the paragraph following Theorem 2.6, we remarked that the verbatim counterpart of unboundedness condition (a.k.a First Brauer-Thrall Conjecture) holds for brick infinite algebras. However, we warn the reader that for brick infinite algebras, the analogue of the strongly unboundedness statement (a.k.a Second Brauer-Thrall Conjecture) is wrong (For example, let Λ be the Kronecker algebra.). Although the First and Second Brauer-Thrall conjectures were proved more than 30 years ago (respectively, in 1968 and 1984), and consequently the focus of research was shifted from the study of min-rep-inf algebras to some other problems, the interest in these algebras continued beyond their contributions to the proofs of these fundamental conjectures. In fact, the explicit classification of some families of these algebras became available only very recently. For more details, see [Bo1, Bo2, Bo3], [Ri], [Sk], and the references therein.

Motivated by the classification of min-rep-inf algebras recalled in Section 1, it is natural to ask for an analogous classification of minimal brick infinite algebras in terms of their quivers and relations. In particular, one may wish to see an explicit family of minimal brick infinite algebras which are not min-rep-inf. In [Mo1], the first-named author slightly generalized the family of algebras treated in [Ri], and introduced "generalized barbell algebras". This gives an infinite family of gentle algebras which are minimal brick infinite but there is a proper quotient of them which is min-rep-inf. In fact, that proper quotient is called a "windwheel algebra" and are shown to be brick-finite (see [Ri] and [Mo1]).

In our future work, we give a concrete classification of those minimal brick infinite algebras which are (special) biserial. Namely, we produce the analogue of the recent results of Ringel [Ri], in which he fully described those minimal representation infinite algebras which are (special) biserial. Our new results allow us to prove a stronger version of our main conjectures in Section 1. More specifically, we can study brick infiniteness of biserial algebras via existence of certain generic modules.

We recall that an indecomposable Λ-module G is *generic* if it is of infinite length as a Λ-module, but of finite endolength. Namely, the length of G, viewed as module over $\mathrm{End}_\Lambda(G)$, is finite. Consequently, by *generic brick* we mean a generic module whose endomorphism algebra is a division ring. As shown in [C-B], generic modules play a fundamental role in the study of representation infinite algebras, and in particular the tame ones. Thanks to the intuition we have developed through our study of biserial algebras, we believe that the notion of "generic brick" can provide fresh impetus to the study of our conjectures 1.1 and 1.2, particularly in the study of bricks over tame algebras.

Similarly, it is reasonable to search for the full classification of those minimal brick infinite algebras which are non-distributive. That is to describe the counterpart of the family $\mathrm{Mri}(\mathfrak{F}_{\mathrm{nD}})$ studied in [Bo2]. Particularly, we observe that in Theorem 2.7, the situation in the last assertion of the theorem can occur only for non-distributive algebras. Hence, it is natural to ask whether it is true in general that a minimal brick infinite algebra Λ is non-distributive if and only if $\mathrm{brick}(\Lambda)$ contains an infinite family of unfaithful bricks. We remark that the analogous classification holds for min-rep-infinite algebras. That is, a min-rep-inf algebra Λ is non-distributive if and only if $\mathrm{ind}(\Lambda)$ contains an infinite family of unfaitfhul modules (for more details, see [Ja], [Ku] and [Bo2]).

References

[Ad] T. Adachi, *Characterizing τ-tilting finite algebras with radical square zero*, Proc. Amer., 4673-4685, (2016).

[Au] M. Auslander, *Large modules over artin algebras*, Algebra, Topology and Category Theory: a collection of papers in honor of Samuel Eilenberg; Academic Press (1976), 1–17.

[AIR] T. Adachi, O. Iyama, I. Reiten, *τ-tilting theory*, Compos. Math. 150 (2014), no. 3, pp 415-452.

[ASS] I. Assem, D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras*, Volume 1, Cambridge University Press, Cambridge, 2006.

[Ba] R. Bautista, *On algebras of strongly unbounded representation type*, Comment. Math. Helv. 60(1985), 392-399.

[Bo1] K. Bongartz, *Indecomposables live in all smaller lengths*, Represent. Theory 17 (2013), 199-225.

[Bo2] K. Bongartz, *On minimal representation infinite algebras*, arXiv:1705.10858v4.

[Bo3] K. Bongartz, *On representation-finite algebras and beyond*, Advances in representation theory of algebras, 65–101, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, (2013).

[BB] S. Brenner, M. C. R. Butler, *Generalisations of the Bernstein– Gelfand–Ponomarev reflection functors*, Proc. ICRA II (Ottawa, 1979), Lecture Notes in Math. No. 832, Springer-Verlag, Berlin, Hei- delberg, New York, 1980, pp. 103–69.

[BCZ] E. Barnard, A. Carroll and S. Zhu, *Minimal inclusions of torsion classes*, Algebraic Combinatorics, Volume 2 (2019) no. 5, p. 879-901.

[BG+] R. Bautista, P. Gabriel, A.V. Roiter, L. Salmerón, *Representation-finite algebras and multiplicative bases*, Inventiones mathematicae 81, 217-285, (1985).

[C-B] W. Crawley-Beovey, *Tame algebras and generic modules*, Proceedings of the London Mathematical Society (1991), 241–265.

[CKW] C. Chindris, R. Kinser, J. Weyman, *Module Varieties and Representation Type of Finite-Dimensional Algebras*, International Mathematics Research Notices, Volume 2015, Issue 3, 2015, 631–650.

[DIJ] L. Demonet, O. Iyama, G. Jasso, *τ-tilting finite algebras and g-vectors*, Int. Math. Res. Not. IMRN (2019), 852–892.

[DI+] L. Demonet, O. Iyama, N. Reading, I. Reiten, H. Thomas, *Lattice theory of torsion classes*, arXiv:1711.01785v2, 2018.

[GL+] F. Geiss, D. Labardini-Fragoso, J. Schröer *Schemes of modules over gentle algebras and laminations of surfaces*, arXiv:2005.01073, 2020.

Schemes of modules over gentle algebras and laminations of surfaces

[Ja] J. P. Jans, *On the indecomposable representations of algebras*, Annals of Mathematics 66 (1957), 418-429.

[KPY] B. Keller, P.-G. Plamondon and T. Yurikusa, *Tame algebras have dense g-vector fans*, IMRN, 2021, published online.

[Ku] H. Kupisch, *Symmetrische Algebren mit endlich vielen unzerlegbaren Darstellungen*, (German) J. Reine Angew. Math. 219, 1–25, (1965).

[Mo1] K. Mousavand, *τ-tilting finiteness of biserial algebras*, arXiv:1904.11514.

[Mo2] K. Mousavand, *τ-tilting finiteness of non-distributive algebras and their module varieties*, arXiv:1910.02251.

[MP] K. Mousavand, C. Paquette *Minimal τ-tilting infinite algebras*, arXiv:2103.12700.

[Pl] P-G. Plamondon, *τ-tilting finite gentle algebras are representation-finite*, Pacific Journal of Mathematics 302(2):709-716, (2019).

[Ri] C.M. Ringel, *The minimal representation infinite algebras which are special biserial.*, Representations of Algebras and Related Topics, EMS Series of Congress Reports, European Math. Soc. Publ. House, Zürich, 2011.

[Sk] A. Skowroński, *Minimal representation-infinite Artin algebras*, Math. Proc. Camb. Phil. Soc., 116, 229-243, (1994).

[ST] S. Schroll, H. Treffinger, *A τ-tilting approach to the first Brauer-Thrall conjecture*, arXiv:2004.14221v3.

[STV] S. Schroll, H. Treffinger, Y. Valdivieso, *On band modules and τ-tilting finiteness*, arXiv:1911.09021.

[Se] F. Sentieri, *A brick version of a theorem of Auslander*, arXiv:2011.09253.

[SS] D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras*, Volume 2, Cambridge University Press, Cambridge, 2007.

Department of Mathematics and Statistics, Queen's University, Kingston ON, Canada
Email address: mousavand.kaveh@gmail.com

Department of Mathematics and Computer Science, Royal Military College of Canada, Kingston ON, Canada
Email address: charles.paquette.math@gmail.com