STRICT DIHEDRAL ALGEBRAS ARE τ-TILTING FINITE

Abstract

In these notes we show that every algebra of strict dihedral type is τ-tilting finite. This family of algebras, which originally arise from triangulations of surfaces, gives a new concrete family of representation-infinite algebras that are τ-tilting finite. We also compare them with the family of minimal representation-infinite special biserial algebras and observe that they form two disjoint subfamilies of special biserial algebras with some interesting properties in common.

1. Preliminaries

Let k be an algebraically closed field. Λ always denotes a finite dimensional k-algebra and $\operatorname{mod}\Lambda$ is the category of all finitely generated left Λ-modules. Every quiver $Q=(Q_0,Q_1,s,e)$ is a directed graph, where the vertex set Q_0 and the arrow set Q_1 are assumed to be finite. By $\mathfrak{l}(\Lambda)$ we denote the number of isomorphism classes of simple modules in $\operatorname{mod}\Lambda$, or, equivalently, the rank of the Grothendieck group $K_0(\Lambda)$ of Λ. Provided that $\Lambda = kQ/I$, where I is an admissible ideal in kQ, we have $\mathfrak{l}(\Lambda)=|Q_0|$. In this case (Q,I) denotes the bound quiver of Λ.

1.1. **Algebras of strict dihedral type.** Building upon [E1] and [E2], in [ES1] an algebra Λ with $\mathfrak{l}(\Lambda)\geq 2$ is called *generalized dihedral type* if it satisfies the following properties:

(1) Λ is symmetric, indecomposable and tame.
(2) The stable Auslander-Reiten quiver $\Gamma^s(\Lambda)$ consists of the following compoenents:
   (i) stable tubes of ranks 1 and 3;
   (ii) at least one non-periodic components of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$ or $\mathbb{Z}\widetilde{\mathbb{A}}_n$;
(3) Ω_Λ fixes all stable tubes of rank 3 in $\Gamma^s(\Lambda)$.

Suppose $\mathfrak{F}_{\mathrm{gD}}$ denotes the family of all algebras of generalized dihedral type. Then, by the second condition, it is obvious that every Λ in $\mathfrak{F}_{\mathrm{gD}}$ is representation-infinite algebras. In fact, in [ES1], it is shown that $\mathfrak{F}_{\mathrm{gD}}\subset\mathfrak{F}_{\mathrm{B}}$, where by $\mathfrak{F}_{\mathrm{B}}$ we denote the family of all biserial algebras. From the defining conditions of algebras of generalized dihedral type it is evident that there are biserial algebras which are not of generalized diheral type, thus the aforementioned containment is strict. For a geometric characterization of the algebras $\mathfrak{F}_{\mathrm{gD}}$ in terms of weighted surface algebras, see [ES1].

To avoid confusion, we should remark that the notion of algebra of generalized dihedral type does not extend the family of algebras of dihedral type (as defined in [E2]) to that of $\mathfrak{F}_{\mathrm{gD}}$. In particular, suppose $\mathfrak{F}_{\mathrm{D}}$ denotes the family of algebras of dihedral type, as defined in [E2]. Due to the complete description of the algebras in $\mathfrak{F}_{\mathrm{D}}$ in terms of their bound quivers, one should note that there exists an algebra Λ in $\mathfrak{F}_{\mathrm{D}}$ with $\mathfrak{l}(\Lambda)=1$. This obviously shows that $\mathfrak{F}_{\mathrm{D}}\not\subseteq\mathfrak{F}_{\mathrm{gD}}$.

An algebra Λ in $\mathfrak{F}_{\mathrm{gD}}$ is called of *strict dihedral type* if it additionally satisfies the following conditions:

(4) $\mathfrak{l}(\Lambda) \leq 3$.
(5) $\Gamma^s(\Lambda)$ has $\mathfrak{l}(\Lambda) - 1$ stable tubes of rank 3.
(6) the Cartan matrix C_Λ is non-singular.

Assume $\mathfrak{F}_{\mathrm{sD}}$ denotes the family of all algebras of strict dihedral type. As mentioned in [E1], if Λ is a non-local block of a group algebras and the defect group of Λ is dihedral, then Λ belongs to $\mathfrak{F}_{\mathrm{sD}}$.

If $\mathfrak{F}_{\mathrm{sB}}$ denotes the family of special biserial algebras, then we have the following strict containments of the families of algebras we are interested in

$$\begin{array}{ccc} \mathfrak{F}_{\mathrm{sD}} & \subset & \mathfrak{F}_{\mathrm{gD}} \\ \cap & & \cap \\ \mathfrak{F}_{\mathrm{sB}} & \subset & \mathfrak{F}_{\mathrm{B}} \end{array}$$

The following theorem of Erdmann and Skowroński [ES1] shows that the subfamily $\mathfrak{F}_{\mathrm{sD}}$ inside the family of all algebras of generalized dihedral type could be fully characterized in terms of the Cartan matrix C_Λ of algebras Λ in $\mathfrak{F}_{\mathrm{gD}}$.

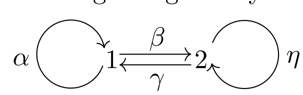
Theorem 1.1 *(*[ES1, Theorem 2]*) Let Λ be an algebra of generalized dihedral type. Then, Λ is of strict dihedral type if and only if C_Λ is non-singular.*

The above theorem immediately implies that for every Λ in $\mathfrak{F}_{\mathrm{gD}}$ with $\mathfrak{l}(\Lambda) > 3$, the Cartan matrix C_Λ is always singular.

Furthermore, a full description of the algebras in $\mathfrak{F}_{\mathrm{sD}}$ in terms of their bound quivers is given in [ES1], which we recall in the following. To do so, following Erdmann and Skowroński, we first list three types of bound quivers that arise from triangulations of weighted surface. For the most part, we follow the notation used in [ES1] and [ES2]. However, we warn the reader that our order of composition of arrows is opposite to that used in the aforementioned papers. In particular, if α and β are two arrows in a quiver Q such that α ends where β starts, we write $\beta\alpha$ to denote the path of length two which starts at $s(\alpha)$, first goes through α and then through β, and it ends at $e(\beta)$.

For the geometric origin and motivations behind the name associated to each type of the following bound quivers, see [ES1] and [ES2].

Disc Type. Let $\Lambda(r, s, b)$ be the algebra given by the quiver



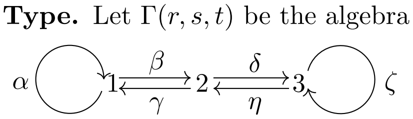
subject to the following relations

$$\beta\alpha = \alpha\gamma = \gamma\beta = 0$$

$$\alpha^r = (\gamma\eta\beta)^s, \quad (\eta\beta\gamma)^s = (\beta\gamma\eta)^s, \quad \eta^2 = b(\beta\gamma\eta)^s,$$

where r and s are positive integers and $b \in k$.

Two-sum projective Type. Let $\Gamma(r, s, t)$ be the algebra given by the quiver

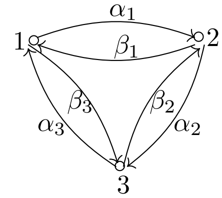


subject to the following relations

$$\alpha\gamma = \beta\alpha = \gamma\beta = \zeta\delta = \eta\zeta = \delta\eta = 0$$
$$\alpha^r = (\gamma\eta\delta\beta)^s, \quad (\eta\delta\beta\gamma)^s = (\beta\gamma\eta\delta)^s, \quad \zeta^t = (\delta\beta\gamma\eta)^s,$$

where r, s and t are positive integers.

Triangle Type. Let $\Omega(a, b, c)$ be the algebra given by the quiver



subject to the following relations

$$\alpha_2\alpha_1 = \alpha_3\alpha_2 = \alpha_1\alpha_3 = 0 = \beta_3\beta_1 = \beta_2\beta_3 = \beta_1\beta_2$$
$$(\beta_1\alpha_1)^a = (\alpha_3\beta_3)^c, \quad (\beta_2\alpha_2)^b = (\alpha_1\beta_1)^a \quad (\beta_3\alpha_3)^c = (\alpha_2\beta_2)^b,$$

where a, b and c are positive integers.

Using the above terminology, now we can state an important result of [ES1] on the classification of algebras in the family $\mathfrak{F}_{\mathrm{sD}}$.

Theorem 1.2 *(*[ES1, Theorem 8.5]*) An algebra $\Lambda = kQ/I$ is of strict dihedral type if and only if its bound quiver (Q, I) is disc or two-sum projective or triangle.*

1.2. **morphisms between string modules.** Because our methodology heavily relies on the analysis of the morphism between string modules over special biserial algebras, we briefly recall the notion of graph maps, introduced in [CB1] and [S]. As shown in the aforementioned papers, for every pair of strings v and w in a special biserial algebra, graph maps form a concrete basis for the space $\mathrm{Hom}(M(w), M(v))$.

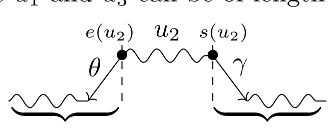
Definition 1.3 If $\Lambda = kQ/I$ is a special biserial algebra, by $\mathrm{Str}(\Lambda)$ we denote the set of strings in the bound quiver (Q, I). Moreover, for each $u \in \mathrm{Str}(\Lambda)$, the set of *factorizations of* u is given by

$$\mathtt{F}(u) := \{(u_3, u_2, u_1) \mid u_3, u_2, u_1 \in \mathrm{Str}(\Lambda) \text{ and } u = u_3u_2u_1\}.$$

For every $(u_3, u_2, u_1) \in \mathtt{F}(u)$, let $(u_3, u_2, u_1)^{-1} := (u_1^{-1}, u_2^{-1}, u_3^{-1}) \in \mathtt{F}(u^{-1})$. If $(u_3, u_2, u_1) \in \mathtt{F}(u)$, then it is called a *quotient factorization* of u if

(i) $u_1 = e_{s(u_2)}$ or $u_1 = \gamma^{-1}u_1'$ with γ in Q;
(ii) $u_3 = e_{e(u_2)}$ or $u_3 = u_3'\theta$ with θ in Q.

From each quotient factorization $(u_3, u_2, u_1) \in \mathtt{F}(u)$, we get a quotient morphism of Λ-modules from $M(u)$ to $M(u_2)$. By $\mathtt{F_q}(u)$ we denote the set of all quotient factorizations of u. The general configuration of every element of $\mathtt{F_q}(u)$ can be visualized as follows, where u_1 and u_3 can be of length zero.



Dual to the above notion, $(u_3, u_2, u_1) \in \mathtt{F}(u)$ is called a *submodule factorization* of u provided that

(i) $u_1 = e_{s(u_2)}$ or $u_1 = \gamma u_1'$ with γ in Q;
(ii) $u_3 = e_{e(u_2)}$, or $u_3 = u_3'\theta^{-1}$ with θ in Q,

and by $\mathtt{F_s}(u)$ we denote the set of all submodule factorizations of u. Similarly, every (u_3, u_2, u_1) in $\mathtt{F_s}(u)$ induces an inclusion of the associated Λ-modules, from $M(u_2)$ into $M(u)$. Every such inclusion can also be visualised by a pair of diagrams dual to the one illustrated above.

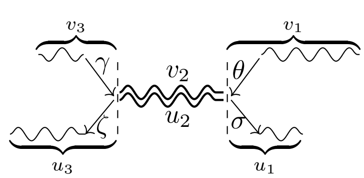
Definition 1.4 Let Λ be a special biserial algebra and $u, v \in \mathrm{Str}(\Lambda)$. Then, we say a pair $((u_3, u_2, u_1), (v_3, v_2, v_1)) \in \mathtt{F_q}(u) \times \mathtt{F_s}(v)$ is *admissible* if $u_2 = v_2$ or $u_2 = v_2^{-1}$. The collection of all admissible pairs in $\mathtt{F_q}(u) \times \mathtt{F_s}(v)$ is denoted by $\mathtt{A}(u, v)$. For each $T = ((u_3, u_2, u_1), (v_3, v_2, v_1))$ in $\mathtt{A}(u, v)$ consider f_T defined as follow: f_T is the composition of the projection $M(u) \to M(u_2)$, followed by the identification of $M(u_2)$ with $M(v_2)$, followed by the inclusion $M(v_2) \to M(v)$. The resulting map is a morphism $f_T : M(u) \to M(v)$ in $\mathrm{mod}\,\Lambda$, called the *graph map* given by T. For $w \in \mathrm{Str}(\Lambda)$, an admissible pair T in $\mathtt{A}(w, w)$ is *trivial* if f_T is the identity morphism $id : M(w) \to M(w)$.

The next theorem gives a concrete description of the space of homomorphisms between string modules. In the following, for two strings u and v, we write $u \sim v$, provided that $u = v$ or $u = v^{-1}$.

Theorem 1.5 [CB1] *Let Λ be a special biserial algebra and u and v are in* $\mathrm{Str}(\Lambda)$. *The set of graph maps $\{f_T \mid T \in \mathtt{A}(u, v)\}$ forms a basis for* $\mathrm{Hom}_\Lambda(M(u), M(v))$. *Furthermore, the following are equivalent:*

(1) $T = ((u_3, u_2, u_1), (v_3, v_2, v_1))$ is a non-trivial admissible pair in $\mathtt{A}(w, w)$;
(2) There are two distinct substrings u_2 and v_2 of w such that $v_2 \sim u_2$ which satisfy $w = u_3u_2u_1$ and $w = v_3v_2v_1$ with $l(u_1)+l(u_3) > 0$ and $l(v_1)+l(v_3) > 0$, where u_1 and u_3 (respectively v_1 and v_3) leave u_2 (respectively enter v_2).

The second part of the above theorem asserts that for each non-trivial endomorphism of a string module $M(w)$, there must exist a proper substring of w which occurs in w at least twice, once on the top w and another time at the bottom of it. Here, by the *top* and *bottom* of w we respectively refer to the local configuration of u_2 and v_2 in an admissible pair $T = ((u_3, u_2, u_1), (v_3, v_2, v_1))$ in $\mathtt{A}(w, w)$, which is illustrated in the following. Note that every such configuration gives rise to a (non-identity) graph map in $\mathrm{End}_\Lambda(M(w))$:



$$\begin{array}{c} M(w) = M(v_3v_2v_1) \\ \uparrow \iota \\ M(u_2) \simeq M(v_2) \\ \uparrow \pi \\ M(w) = M(u_3u_2u_1) \end{array}$$

In the above picture, the substrings u_i's and v_i's, for every $1 \leq i \leq 3$, can be of any length, provided the satisfy $l(u_1) + l(u_3) > 0$ and $l(v_1) + l(v_3) > 0$. By these inequalities we guarantee that at least one of the two arrows σ or ζ (respectively θ or γ) that leave u_2 (respectively which enter v_2) is actually present in the configuration. Each of the squiggly segments of the substrings (except for the specified arrows), can have any internal configuration of arrows.

1.3. **τ-Tilting theory.** Introduced by Adachi, Iyama and Reiten [AIR], the concept of τ-tilting theory is primarily aimed at resolving the deficiency of tilting theory with respect to mutation. In this subsection we only collect some basic materials we need in order to recall an explicit criterion for the notion of τ-tilting finiteness of algebras. For the motivations and further details on the subject, we refer to [AIR], [DIJ] and [DI+].

Recall that a Λ-module M is called *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$. Let $\tau\text{-rigid}(\Lambda)$ and $\mathtt{i}\tau\text{-rigid}(\Lambda)$) respectively denote the set of basic τ-rigid modules and the indecomposable τ-rigid modules. A τ-rigid module M is *τ-tilting* if $|M| = |\Lambda|$. Furthermore, M is *support τ-tilting* if M is τ-tilting over $A/\langle e\rangle$, for an idempotent e in A. By $\tau\text{-tilt}(\Lambda)$ and $s\tau\text{-tilt}(\Lambda)$ we respectively denote the set of all basic τ-tilting modules and that of all basic support τ-tilting modules in $\mathrm{mod}\,\Lambda$. Consequently, Λ is called τ-tilting finite if $|\tau\text{-tilt}(\Lambda)| < \infty$. From the Auslander-Reiten duality, it is immediate that every τ-rigid module is rigid.

We also recall that M is a brick if $\mathrm{End}_\Lambda(M)$ is a division algebra. It is known that over algebraically closed fields, M is a brick if and only if $\mathrm{End}_\Lambda(M) \simeq k$. In fact, over an arbitrary field k, by the explicit construction of graph maps between string modules, it is easy to show that a string module X is a brick if and only if $\mathrm{End}_\Lambda(X)$ is isomorphic to k. Let $\mathrm{brick}(\Lambda)$ denote the set of bricks in $\mathrm{mod}\,\Lambda$. From the definition it is obvious that $\mathrm{brick}(\Lambda)$ consists of indecomposable modules.

A full subcategory of $\mathrm{mod}\,\Lambda$ is a *torsion class* if it is closed under quotient and extension. A torsion class $\mathcal{T}$ is *functorially finite*, in the sense of [AR1], if $\mathcal{T} = \mathrm{Gen}(M)$, for some Λ-module M. By $\mathrm{tors}(\Lambda)$ we denote the set of all torsion classes in $\mathrm{mod}\,\Lambda$, while $\text{f-tors}(\Lambda)$ is the set of all functorially finite torsion classes. In a subcategory $\mathcal{C}$ of $\mathrm{mod}\,\Lambda$, a module $X \in \mathcal{C}$ is said to be Ext-*projective* if $\mathrm{Ext}^1_A(X, -)|_{\mathcal{C}} = 0$.

The following theorem establishes a connection between the above notions.

Theorem 1.6 ([AIR, Theorem 2.7]) *For each algebra Λ, there is a bijection between $s\tau\text{-tilt}(\Lambda)$ and $\text{f-tors}(\Lambda)$. In particular, in one direction, each basic support τ-tilting module X is sent to $\mathrm{Gen}(X)$. In the other direction, every $\mathcal{T} \in \text{f-tors}(\Lambda)$ is sent to $X_{\mathcal{T}} = \bigoplus X_i$, where the direct sum runs over the isomorphism classes of all* Ext-*projective indecomposable modules X_i in $\mathcal{T}$.*

Now that for an algebra Λ we have seen a correspondence between support τ-tilting modules and the functorially finite torsion classes of $\mathrm{mod}\,\Lambda$, we present some important equivalent conditions that describe τ-tilting finiteness of Λ. Through the following theorem, we view the notion of τ-tilting finiteness as a natural generalization of the notion of representation-finiteness of algebras. This is because, as shown in [AR1], an algebra Λ is representation-finite if and only if every (full) subcategory of $\mathrm{mod}\,\Lambda$ is functorially finite. The study of τ-tilting finiteness of algebras from thie viewpoint has been already exploited for the family (special) biserial algebras (for details, see [Mo]).

Theorem 1.7 ([DIJ, DI+]) *For an algebra Λ, the following are equivalent.*

(1) Λ *is τ-tilting finite;*
(2) $\text{f-tors}(\Lambda) = \mathrm{tors}(\Lambda)$*;*
(3) $\mathrm{tors}(\Lambda)$ *is finite;*
(4) $\mathtt{i}\tau\text{-rigid}(\Lambda)$ *is finite;*
(5) $\mathrm{brick}(\Lambda)$ *is finite.*

Let us finish this subsection by recalling an important result in the τ-tilting theory from the lattice theoretical point of view. In particular, in [Mo], the author uses a more elementary version of the following statement to reduce the problem of τ-tilting infiniteness of biserial algebras to the min-rep-infinite biserial algebras, which are significantly more tractable.

Theorem 1.8 ([DI+]) *Every surjective morphism $\phi : \Lambda_1 \to \Lambda_2$ of algebras induces a surjective lattice map $\tilde{\phi} : \mathrm{tors}(\Lambda_1) \to \mathrm{tors}(\Lambda_2)$, defined by $\tilde{\phi}(\mathcal{T}) := \mathcal{T} \cap \mathrm{mod}\,\Lambda_2$, for each $\mathcal{T} \in \mathrm{tors}(\Lambda_1)$. In particular, if Λ_1 is τ-tilting finite, so is Λ_2.*

2. Main Results

The main theorem of these notes is the following:

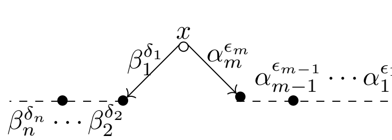
Theorem 2.1 *Every algebra of strict dihedral type is τ-tilting finite.*

To prove the above assertion, by Theorem 1.7, it suffices to show that every algebra of strict dihedral type admits only finitely many isomorphism classes of bricks. For the sake of readability, we prove the above theorem via a sequence of lemmas. To simplify the proofs, let us first fix some terminology in the following definition.

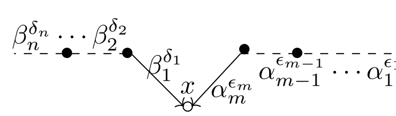
Definition 2.2 Let kQ/I be a special biserial algebra and u and v be strings in (Q, I). Suppose $u = \alpha_m^{\epsilon_m} \cdots \alpha_2^{\epsilon_1} \alpha_1^{\epsilon_1}$ and $v = \beta^{\delta_n} \cdots \beta_2^{\delta_2} \beta_1^{\delta_1}$, for some $\alpha_i, \beta_j \in Q_1$ and $\epsilon_i, \delta_j \in \{\pm 1\}$, for every $1 \leq i \leq m$ and $1 \leq j \leq n$.

(1) For an arrow $\gamma \in Q_1$, the string u *supports* γ provided $\alpha_i^{\epsilon_i} = \gamma$ or $\alpha_i^{\epsilon_i} = \gamma^{-1}$, for some $1 \leq i \leq p$. For a vertex $x \in Q_0$, we say u *visits* x if $s(\alpha_i) = x$ or $e(\alpha_i) = x$, for some $1 \leq i \leq m$.
(2) u *passes through* x if $x = e(\alpha_j^{\epsilon_j}) = s(\alpha_{j+1}^{\epsilon_{j+1}})$, for some $1 \leq j < m$. In this case, we say x is a *plain* vertex of u if $\epsilon_i = \epsilon_{i+1}$. Otherwise, u *changes direction* at x.
(3) We say *tail of u collides with head of v* if $e(u) = s(v)$ and $\epsilon_m = -\delta_1$, but $\alpha_m^{\epsilon_m} \neq \beta_1^{-\delta_1}$. Then, the ordered pair (u, v) is called *tail-head collision*. Moreover, such a tail-head collision pair (u, v) is called an *up-down collision* if $\epsilon_m = -1$ and *down-up collision* otherwise.

The following configurations illustrate a tail-head collision pair (u, v) at the vertex $x = e(u) = s(v)$, where the string $w := vu$ changes direction. In particular, the left picture is an up-down collision, whereas the right picture is a down-up collision. We remark that the length and orientation of the dashed segments could be anything and they may also share some arrows or vertices.



Up-down collision: $\epsilon_m = -1$ and $\delta_1 = 1$.



Down-up collision: $\epsilon_m = 1$ and $\delta_1 = -1$.

Now that we have the required terminology and tools at our disposal, we outline our main strategy. The rest of this section is mainly devoted to showing that for every algebra Λ of strict dihedral type, $\mathrm{brick}(\Lambda)$ is always finite, which itself, by Theorem 1.7, is equivalent to τ-tilting finiteness of Λ. To accomplish this goal, recall that every algebra of strict dihedral type is special biserial and by Theorem 1.2, the bound quiver of every such algebra Λ is fully described. Hence, we employ the methodology of [Mo] for the the study of τ-tilting finiteness of (special) biserial algebras.

It is known that if Λ is special biserial, every indecomposable Λ-module M is either a string module, or a band module, or M is projective-injective. Because there is only finitely many of the last type of indecomposable modules, we only need to show that for each Λ in $\mathfrak{F}_{\mathrm{sD}}$, almost every string and band Λ-module admits a nontrivial endomorphism which is not invertible. Hence, for an algebras of strict dihedral type, using the explicit description of its bound quiver given in Section 1, first we show that there is a sufficiently large integer d such that for any string $w = \theta_p^{\epsilon_p} \cdots \theta_2^{\epsilon_2} \theta_1^{\epsilon_1}$ in $\mathrm{Str}(\Lambda)$ whose length p is larger than d, the string module $M(w)$ is never a brick. Similarly, we show that if w is a band, every band module associated to that is not a brick either. In fact, for every indecomposable M in $\operatorname{mod}\Lambda$ which is a string or band module, we find a vertex x in the corresponding string w that appears both on the top and in the bottom of w. This implies the simple module S_x is a summand of both the top and the socle of M, and from this we get a non-invertible nonzero endomorphism of the string or band module M, arising from a pair of morphisms $\pi : M(w) \to S_x$ and $\iota : S_x \to M(w)$, which are respectively an epimorphism and a monomorphism of Λ-modules. Through the composition $\iota \circ \pi : M(w) \to M(w)$, we have a nonzero map in $\mathrm{End}_\Lambda(M(w))$ which is not invertible. This implies that for almost all Λ-modules M, we have $\dim_k(\mathrm{End}_\Lambda(M)) \geq 2$ and therefore $\mathrm{brick}(\Lambda)$ is finite.

Remark 2.3 In each of the following proofs, depending on the bound quiver of the algebra we treat, one can potentially find a precise linear formula for the boundary used for the length of strings. However, since we are primarily interested in the problem of finiteness of string and band modules that are brick, we do not specify the smallest boundary that could be used, and instead give a sufficiently large number in terms of a non-linear equation given by the integers that come with the bound quivers.

We start by analyzing the brick finiteness of the the simplest subfamily of algebras of strictly dihedral type, which consists of the algebras in $\mathfrak{F}_{\mathrm{sD}}$ whose Grothendieck group is of rank 2.

Lemma 2.4 *If Λ is an algebra of disc type, then it is brick-finite.*

Proof. From Theorem 1.2, we know that Λ is of the form $\Lambda(r, s, b)$, for some integers $r > 1$ and $s > 0$ and some b in k, as in the previous section.

First we show that for every string $w = \theta_p^{\epsilon_p} \cdots \theta_2^{\epsilon_2} \theta_1^{\epsilon_1}$ in $\mathrm{Str}(\Lambda(r, s, b))$, if p is sufficiently large, the string module $M(w)$ is not a brick (for example, assume $p > 4(r+s)^{r+s}$).

We claim that if w supports α, vertex 1 appears both on the top and in the bottom of w, hence $M(w)$ admits a nontrivial endomorphism given by the composition $M(w) \twoheadrightarrow S_1 \hookrightarrow M(w)$, which passes through the simple module S_1 and is not invertible. To see this, let u be a maximal sequence of α (or α^{-1}) which appear in w. Namely, u is a substring of w of the form $u = \alpha^d$, for some nonzero integer d that is maximal with this property. Then, $s(u) = e(u) = 1$ and one of these two vertices is necessarily on the top and the other one is in the bottom of $M(w)$.

Now we assume w does not support α. Moreover, from our assumption it is immediate that w supports each of the arrows β, γ and η and passes through both vertices 1 and 2. Because w does not support α, vertex 1 can only appear as transition vertex of w. However, the relations in the bound quiver and the length of the string w implies that w must change the direction, which occurs at 2. This puts S_2 either in the socle or top of $M(w)$. Without loss of generality, suppose the first change of direction puts S_2 in the socle of $M(w)$. Then, due to the length of w, it is easy to see there must also exists another change of direction in w which occurs at 2 and puts S_2 in the top of $M(w)$ and we get the desired result.

The same argument shows that every band in $\mathrm{Str}(\Lambda)$ also admits a nonzero endomorphism which is not invertible.

□

Now we analyze the brick-finiteness of the rest of algebras in $\mathfrak{F}_{\mathrm{sD}}$. The next lemma addresses the algebras of the two-sum projective type, as defined in the previous section.

Lemma 2.5 *Suppose Λ is an algebra of two-sum projective type. Then Λ is brick-finite.*

Proof. By Theorem 1.2, Λ is isomorphic to an algebra of the form $\Gamma(r,s,t)$, for some positive integers r,s and t.

Let $w = \theta_p^{\epsilon_p} \cdots \theta_2^{\epsilon_2}\theta_1^{\epsilon_1}$ be in $\mathrm{Str}(\Gamma(r,s,t))$ whose length is sufficiently large, say for example $p > 5(r+s+t)^{r+s+t}$. We show that $M(w)$ is not a brick.

First we remark that, due to the relations of the bound quiver of $\Gamma(r,s,t)$, the same argument as in Lemma 2.4 shows that if w supports α (respectively ζ), then vertex 1 (respectively 3) appears both on the top and in the bottom of w. Hence, we restrict to the case where w does not support α nor ζ and show that there still exists a vertex that occurs both on the top and in the bottom of w.

Analogous to the second part of the proof of Lemma 2.4, since w does not support α and ζ, vertices 1 and 3 can appear only as the plain vertices of w. Moreover, because of the relations in the bound quiver and the length of w, one observes that w must change direction, which must occur at vertex 2 and this puts S_2 in the socle or top of $M(w)$. Furthermore, because the length of w is chosen sufficiently large, the relations implies that in fact w changes the direction at least twice which happens at vertex 2 and puts S_2 both in the socle and top of $M(w)$ and gives the desired result.

As for the band modules over Λ, note that if w is a band in $\mathrm{Str}(\Lambda)$, due to the relations in the bound quiver of $\Gamma(r,s,t)$, we observe that w must visit all vertices. Hence, the same argument as above shows that the associated band modules admit a nonzero endomorphism which is not invertible. This finishes the proof.

□

In the next lemma, we treat the brick-finiteness of the last type of algebras of strict dihedral type.

Lemma 2.6 *If Λ is an algebra of triangle type, then it is brick-finite.*

Proof. By Theorem 1.2, Λ is of the form $\Omega(a,b,c)$, for some positive integers a,b and c. Suppose $w = \theta_p^{\epsilon_p} \cdots \theta_2^{\epsilon_2}\theta_1^{\epsilon_1}$ in $\mathrm{Str}(\Lambda(r,s,b))$ such that p is sufficiently large (for instance, assume $p > 4(a+b+c)^{a+b+c}$). We show that the string module $M(w)$ is not a brick.

First note that, due to the relations of the bound quiver and the assumption on the length of the string, w changes direction at every vertex of $\Omega(a,b,c)$ (therefore it passes through all vertices). Because the local configuration at every vertex is similar, we only prove that for the string w if $s(w)=1$, then $M(w)$ has a nontrivial endomorphism which is not invertible. Given that $s(w)=1$, then $\theta_1^{\epsilon_1}$ belongs to the set $\{\alpha_1, \beta_3, \beta_1^{-1}, \alpha^{-1}\}$.

If $\theta_1^{\epsilon_1} = \alpha_1$, vertex 1 is on the top of w. If 1 also appears in the bottom of w, we are done. Thus we assume otherwise. This implies that for any other $\theta_i^{\epsilon_i}$ with $e(\theta_i^{\epsilon_i}) = 1$, vertex 1 must occur either as a plain vertex of w or on the top of w. Namely, for every $1 < i \leq p$ with $e(\theta_i^{\epsilon_i}) = 1$, we either have a plain at 1 (being $\theta_{i+1}^{\epsilon_{i+1}}\theta_i^{\epsilon_i} = \alpha_1\beta_1$ or $\theta_{i+1}^{\epsilon_{i+1}}\theta_i^{\epsilon_i} = \beta_3\alpha_3$), or otherwise w changes direction via an up-down collision at 1 (being $\theta_{i+1}^{\epsilon_{i+1}}\theta_i^{\epsilon_i} = \alpha_1\beta_3^{-1}$ or $\theta_{i+1}^{\epsilon_{i+1}}\theta_i^{\epsilon_i} = \beta_3\alpha_1^{-1}$).

Suppose $1 < j < p$ is such that $e(\theta_j^{\epsilon_j}) = 3$ and j is minimal with this property. Hence, the substring $\theta_{j-1}^{\epsilon_{j-1}} \cdots \theta_2^{\epsilon_2}\theta_1^{\epsilon_1}$ of w only visits vertices 1 and 2, and does not support any arrow from the set $\{\alpha_2, \alpha_3, \beta_2, \beta_3\}$. Moreover, $\theta_{j-1}^{\epsilon_{j-1}} \cdots \theta_2^{\epsilon_2}\theta_1^{\epsilon_1}$ does not change direction, thus every vertex that it passes thorough is a plain vertex (note that we may have $j-1=1$, meaning that $\theta_{j-1}^{\epsilon_{j-1}} \cdots \theta_2^{\epsilon_2}\theta_1^{\epsilon_1}$ visits 1 and 2 but does not pass through any vertex).From this, observe that the substring $\theta_j^{\epsilon_j}\theta_{j-1}^{\epsilon_{j-1}} \cdots \theta_2^{\epsilon_2}\theta_1^{\epsilon_1}$ of w has exactly one change of direction at vertex $e(\theta_{j-1}^{\epsilon_{j-1}})$. We conclude $\theta_j^{\epsilon_j} \neq \alpha_3^{-1}$: because if $\theta_j^{\epsilon_j} = \alpha_3^{-1}$, then by minimality of j we should get $\theta_j^{\epsilon_j}\theta_{j-1}^{\epsilon_{j-1}} = \alpha_3^{-1}\beta_1$, which puts 1 in the bottom of w and contradicts our assumption.

Because $\theta_1^{\epsilon_1} = \alpha_1$, $\theta_j^{\epsilon_j} \neq \alpha_3^{-1}$, and $\theta_{j-1}^{\epsilon_{j-1}} \cdots \theta_2^{\epsilon_2}\theta_1^{\epsilon_1}$ of w does not visit 3, from the quadratic relations $\beta_3\beta_1 = 0$ and $\alpha_2\alpha_1 = 0$ (respectively at vertices 1 and 2) we are left with only one choice, being $\theta_j^{\epsilon_j} = \beta_2^{-1}$. In fact we get $\theta_j^{\epsilon_j}\theta_{j-1}^{\epsilon_{j-1}} = \beta_2^{-1}\alpha_1$, which is a down-up collision in w that puts vertex 2 in the bottom of w.

Now, suppose $j < j' < p$ is the smallest integer such that $e(\theta_{j'}^{\epsilon_{j'}}) = 1$. Due to the length of w and the relations of the bound quiver, such j' exists. Note that $\theta_j^{\epsilon_j} = \beta_2^{-1}$. Thus, due to the quadratic relations at vertices 2 and 3, we either have $\theta_{j'}^{\epsilon_{j'}} = \beta_1$ or $\theta_{j'}^{\epsilon_{j'}} = \alpha_3$. The former case (i.e, $\theta_{j'}^{\epsilon_{j'}} = \beta_1$) implies that $\theta_{j'}^{\epsilon_{j'}}\theta_{j'-1}^{\epsilon_{j'-1}} = \beta_1\alpha_2^{-1}$, which is an up-down collision at vertex 2 and puts it on the top of w. Hence, in the substring $\theta_{j'}^{\epsilon_{j'}} \cdots \theta_1^{\epsilon_1}$, vertex 2 appears both on the top and in the bottom, and this gives the desired result. So, suppose $\theta_{j'}^{\epsilon_{j'}} = \alpha_3$, which implies $\theta_{j'}^{\epsilon_{j'}}\theta_{j'-1}^{\epsilon_{j'-1}} = \alpha_3\beta_2^{-1}$ and puts vertex 3 on the top of w.

By a similar argument, there exits $j' < j'' < p$ such that $e(\theta_{j''}^{\epsilon_{j''}}) = 2$. Considering that $\theta_{j'}^{\epsilon_{j'}} = \alpha_3$ and because of the quadratic relations at vertices 1 and 3, the two possible cases are $\theta_{j''}^{\epsilon_{j''}} = \beta_1^{-1}$ or $\theta_{j''}^{\epsilon_{j''}} = \alpha_2^{-1}$. Checking these two cases, one easily concludes that if $\theta_{j''}^{\epsilon_{j''}} = \beta_1^{-1}$, then $\theta_{j''}^{\epsilon_{j''}}\theta_{j''-1}^{\epsilon_{j''-1}} = \beta_1^{-1}\alpha_3$, which puts 1 in the bottom of w. This, along with the fact that $s(w) = 1$ is on the top of w, gives the desired result. Similarly, $\theta_{j''}^{\epsilon_{j''}} = \alpha_2^{-1}$ implies $\theta_{j''}^{\epsilon_{j''}}\theta_{j''-1}^{\epsilon_{j''-1}} = \alpha_2^{-1}\beta_3$, which puts 3 in the bottom of w. This, along with $\theta_{j'}^{\epsilon_{j'}}\theta_{j'-1}^{\epsilon_{j'-1}} = \alpha_3\beta_2^{-1}$, shows that S_3 is both in the socle and top of $M(w)$, so we are done.

The analogous analysis, based on the change of direction the string w must make to respect the relations of the bound quiver, shows that for all other possibilities, being $\theta_1^{\epsilon_1} = \beta_3$, $\theta_1^{\epsilon_1} = \beta_1^{-1}$ and $\theta_1^{\epsilon_1} = \alpha_3^{-1}$, we can always find a vertex appearing both on the top and in the bottom of w, hence the string module $M(w)$ is not a brick. Moreover, from the same argument we conclude that for every band w in $\mathrm{Str}(\Lambda)$, the associated band modules admit nonzero endomorphisms which are not invertible. This completes the proof.

□

Now we prove Theorem 2.1.

Proof of Theorem 2.1. By Theorem 1.2, every algebra Λ of strict dihedral type is of disc type, two-sum projective type or triangle type. By Lemmas 2.4, 2.5 and 2.6, each of these algebras is brick finite. Hence, by Theorem 1.7, Λ is τ-tilting finite. □

The following result is an immediate consequence of Theorem 1.1 and Theorem 2.1.

Corollary 2.7 *Let Λ be an algebra of generalized dihedral type. If the Cartan matrix of Λ is non-singular, then Λ is τ-tilting finite.*

Remark 2.8 Comparing with the new results on τ-tilting finiteness of biserial algebras in [Mo], one notes that Theorem 2.1, along with Theorem 1.2, gives a new family of (special) biserial algebras in terms of their bound quivers which are representation infinite but τ-tilting finite. In the next section we further investigate the algebras of strict dihedral type from the viewpoint of minimal representation-infinite algebras to derive some new results on their module category and their Auslander-Reiten components.

Finally, we note that τ-tilting finiteness of algebras of strict dihedral type provides a deep insight into their module category and their bounded derived category. This is the case because for a τ-tilting finite algebra Λ, by Theorem 1.6, every torsion class $\mathcal{T}$ in $\operatorname{mod}\Lambda$ is functorially finite (i.e, $\mathcal{T} = \mathrm{Gen}(X)$, for some support τ-tilting module X) and the lattice $\mathrm{tors}(\Lambda)$ is finite. Moreover, by [DIJ] one can derive new results on the g-vectors of support τ-tilting modules over every algebra of strict dihedral type. Knowing that these algebras are τ-tilting finite, by [KY] one can also study the silting complexes, t-structures and co-t-structures of every Λ in $\mathfrak{F}_{\mathrm{sD}}$. Additionally, thanks to the explicit bijection between the set of all wide subcategories and the set of torsion classes of every τ-tilting finite algebra (see [MS]), for each Λ in $\mathfrak{F}_{\mathrm{sD}}$ one can further investigate the wide subcategories in $\operatorname{mod}\Lambda$ as well as the lattice structures on $\mathrm{wide}(\Lambda)$ and $\mathrm{tors}(\Lambda)$, which respectively denote the set of all wide subcategories and torsion classes.

3. Quotients of strict dihedral algebras

As remarked in the previous sections, every algebra of generalized dihedral type is biserial (see [ES1, Corollary 3]). Moreover, $\mathfrak{F}_{\mathrm{sD}}$, which denotes the family of all algebras of strict dihedral type, is a subfamily of special biserial algebras $\mathfrak{F}_{\mathrm{sB}}$. Following [Mo], we say a family $\mathfrak{F}$ of k-algebras is *quotient-closed* if for every pair of k-algebras Λ and Λ' and a surjective morphism of algebras $\phi : \Lambda \to \Lambda'$, if Λ belongs to $\mathfrak{F}$, then so does Λ'. From [CB+], we know that the family of all biserial k-algebras is quotient-closed. Furthermore, as shown in [Mo] and it is easy to check, the family of special biserial algebras is also quotient-closed. However, we remark that neither $\mathfrak{F}_{\mathrm{sD}}$ nor $\mathfrak{F}_{\mathrm{gD}}$ is quotient-closed, simply because they all consist of representation-infinite algebras. In fact, thanks to the explicit description of algebras in $\mathfrak{F}_{\mathrm{sD}}$, one observes that if Λ is of strict dihedral type, there exists a representation-infinite quotient Λ' of Λ such that Λ' is not in $\mathfrak{F}_{\mathrm{sD}}$.

We recall that Λ is *minimal representation-infinite* (or min-rep-infinite, for short) if Λ is representation-infinite but every quotient algebra Λ/J is representation-finite, where J is a nonzero two-sided ideal of Λ. Moreover, for a family of k-algebras $\mathfrak{F}$, by $\mathrm{Mri}(\mathfrak{F})$ we denote the family of all min-rep-infinite algebras in $\mathfrak{F}$. As explained in [R2], the study of minimal representation-infinite algebras has played a decisive roles in development of representation theory of algebras in 80's and 90's, in particular in the complete proof of the second Brauer-Thrall conjecture. Among the min-rep-infinite algebras, those which form $\mathrm{Mri}(\mathfrak{F}_{\mathrm{sB}})$ are extensively studied in [R2], primarily because they are of significant interest in various areas of mathematics (e.g. cluster algebras, geometry, combinatorics, mirror symmetry, etc.) and feature the most explicit description between all the min-rep-infinite algebras.

On the other hand, as remarked in [DIJ], the notion of τ-tilting finiteness of algebras can be naturally viewed as a modern generalization of representation-finiteness. Inspired by the conceptual classification of min-rep-infinite algebras recently considered in the work of Bongartz [Bo3] and Ringel [R2], in [Mo] the author introduces the notion of minimal τ-tilting infinite algebras and gives an explicit description of some important families of such algebras in $\mathfrak{F}_{\mathrm{B}}$. Analogous to the classical definition given above, we say Λ is *minimal τ-tilting infinite* if Λ is τ-tilting infinite and every proper quotient of it is τ-tilting finite. Obviously, for a family $\mathfrak{F}$ of algebras, every τ-tilting infinite algebra Λ in $\mathrm{Mri}(\mathfrak{F})$ is minimal τ-tilting infinite. In the case of (special) biserial algebras which closely relate to the scope of this paper (due to the containment $\mathfrak{F}_{\mathrm{sD}} \subsetneq \mathfrak{F}_{\mathrm{sB}}$), in fact in [Mo] it is shown that Λ in $\mathrm{Mri}(\mathfrak{F}_{\mathrm{B}})$ is minimal τ-tilting infinite if and only if Λ is a gentle algebra.

As mentioned at the end of the preceding section, $\mathfrak{F}_{\mathrm{sD}}$ gives another explicit infinite family of rep-infinite special biserial algebras which are τ-tilting finite. In this section, we aim to compare $\mathfrak{F}_{\mathrm{sD}}$ and $\mathrm{Mri}(\mathfrak{F}_{\mathrm{sB}})$, as two interesting subfamilies of special biserial algebras. We should remark that, as explained in the following, these two interesting rep-infinite subfamilies of (special) biserial algebras are disjoint. Therefore, at least at the first glance, it may seem that may not share non-trivial fundamental properties. As we will see, this is not the case. In what follows, we try to verify which of the nice properties that hold for τ-tilting finite members of $\mathrm{Mri}(\mathfrak{F}_{\mathrm{sB}})$ in fact hold for those in $\mathfrak{F}_{\mathrm{sD}}$. To further investigate this comparison, we first recall some terminology.

In order to avoid confusion raised by the difference between the widely used terminology in the literature and that appearing in [HV], following [Mo] we say $\Lambda = kQ/I$ is *weakly minimal representation-infinite* if Λ is rep-infinite but for each vertex $x \in Q_0$, the quotient algebra $\Lambda/\langle e_x \rangle$ is rep-finite. Obviously, every min-rep-infinite algebra is weakly min-rep-infinite algebra but the converse does not hold. We also recall that a component $\mathcal{C}$ of the Auslander-Reiten quiver $\Gamma(\Lambda)$ is called *generalized standard* provided $\mathrm{rad}^{\infty}_{\Lambda}(X,Y) = 0$, for each pair X and Y in $\mathcal{C}$. It is known that a stable tube $\mathcal{T}$ in $\Gamma(\Lambda)$ is generalized standard if and only if $\mathrm{rad}^{\infty}_{\Lambda}(X,Y) = 0$, for every pair of modules on the mouth of $\mathcal{T}$. This itself is known to be equivalent to the fact that the mouth of $\mathcal{T}$ consists of pairwise Hom-orthogonal bricks (for example, see [SS, X.3.3 and X.4.5]).

Now we can state the following proposition on some properties of algebras of strict dihedral type and their rep-infinite quotients.

Proposition 3.1 *Let Λ be an algebra of strict dihedral type and Λ' be a representation infinite quotient of Λ. Then, the following hold:*

(1) Λ is weakly minimal representation-infinite but not minimal representation-infinite.
(2) $\Lambda'/\langle \mathrm{soc}(\Lambda') \rangle$ is a representation-infinite string algebra.
(3) $\Gamma(\Lambda')$ has no preprojective (nor preinjective) component and it has only finitely many generalized standard tubes.

Before proving the above statements we remark that if $\Lambda = kQ/I$ is a self-injective algebra and $\Lambda' = kQ/I'$ is a quotient of Λ, then Λ' needs not to be self-injective. Once again, this implies that unlike the family of (special) biserial algebras, the family of algebras of generalized (similarly strict) dihedral type is not quotient-closed. Hence, the components of the Auslander-Reiten quivers of Λ' do not necessarily satisfy the defining properties of algebras of generalized dihedral type. We wish to also direct the reader to [HV], where Happel and Vossicek give a concrete description of those weakly minimal representation-infinite algebras which admit a preprojective components. By the last part of the above proposition, in fact we claim that if $\Lambda = kQ/I$ and $\Lambda' = kQ/I'$ is as in the third part of the statement, the bound quiver (Q, I') never appears on the list given in [HV].

The following lemma is well-known and plays an important role in the proof of the preceding proposition. For a proof of it we refer to [Mo, Lemma 3.1].

Lemma 3.2 *Let Λ be an algebra and M be a projective-injective module in $\mathrm{mod}\,\Lambda$. Then, Λ and $\Lambda/\langle \mathrm{soc}(M) \rangle$ are of the same representation-type. In particular, a special biserial algebra Λ is representation-infinite if and only if so is $\Lambda/\langle \mathrm{soc}(M) \rangle$.*

Moreover, we recall the following useful criterion for the τ-tilting infiniteness from [Mo].

Lemma 3.3 *For every algebra Λ, if the Auslander-Reiten quiver Γ_Λ of Λ admits a preprojective (similarly preinjective) component, then Λ is τ-tilting infinite.*

Now we can prove the above proposition.

Proof of Proposition 3.1. To be added! □

Analogous to the notation introduced for a family of algebras, if Λ is an algebra, then by $\mathrm{Mri}(\Lambda)$ we denote the family of all isomorphism classes of quotient algebras Λ' of Λ such that Λ' is minimal representation-infinite. Obviously, $\mathrm{Mri}(\Lambda) = \emptyset$ if and only if Λ is represenation-finite.

Theorem 3.4 *Let Λ be an algebra of strict dihedral type. Every Λ' in* $\mathrm{Mri}(\Lambda)$ *is a nody algebra.*

In order to show the above theorem, we need to recall some terminology and results from [Mo].

Definition 3.5 Let $\Lambda = kQ/I$ be an algebra. A vertex x in Q is called a *node* if for every arrow α with $e(\alpha) = x$ and every arrow β with $s(\beta) = x$ we have $\beta\alpha \in I$. Moreover, if $\Lambda = kQ/I$ is minimal representation-infinite special biserial, it is said to be *nody* if (Q, I) has a node.

Furthermore, we have the following result on the τ-tilting finiteness of nody algebras. For the proof, see [Mo, Proposition 5.10].

Proposition 3.6 *Every nody algebra is τ-tilting finite.*

As shown in [Mo, Theorem 6.6], if Λ is a minimal representation-infinite algebra and it is τ-tilting finite, then Λ is either a nody algebra or it is of another type, which is called *wind wheel*. For the definition and properties of the latter family of special biserial algebras, see [R2] and [Mo].

Now we can proof the preceding theorem.

Proof of Theorem 3.4. Suppose $\Lambda' = kQ'/I'$ belongs to $\mathrm{Mri}(\Lambda)$. By Theorem [Mo, Theorem 6.6], (Q', I') is either a wind wheel or a nody bound quiver. We consider all possible cases. Moreover, Theorem 1.2 implies that the algebra Λ is either of disc type, or two-sum projective or of triangle type.

If Λ is of disk type. Then it is isomorphic to $\Lambda(r, s, b)$, for some positive integers r and s and some b in $k \setminus \{0\}$. Because the arrows β and γ are involved in quadratic relations in the bound quiver (Q, I) of $\Gamma(r, s, t)$, they cannot be a bar in the bound quiver of (Q', I'). Hence, they both are present in (Q', I') which implies that Λ' is not a wind wheel algebra. Therefore, Λ' must be a nody algebra.

If Λ is of two-sum projective type. The bound quiver of Λ is given by that of $\Gamma(r, s, t)$ for some positive integers r, s and t. Because the arrows β, γ, δ and η are involved in quadratic relations in the bound quiver (Q, I) of $\Gamma(r, s, t)$, they cannot form a bar in the bound quiver of (Q', I'). If $w \in \mathrm{Str}(\Lambda)$ is a band, we can alsways assume $s(w) = 2$. Moreover, if w supports α (respectively β), then it must support β and γ (respectively δ and η). Hence, the pair β and γ (or similarly δ and η) are simultaneously present in (Q', I') which implies that Λ' cannot be a wind wheel algebra. Hence, (Q', I') is always a nody algebra.

If Λ is of triangle type. In this case we have Λ is isomorphic to an algebra $\Omega(a, b, c)$, for some positive integers a, b and c. Because every arrow of the bound quiver is involved in exactly two quadratic relation, it is easy to see that if (Q', I') is the bound quiver of Λ', we cannot have a bar in (Q', I'), implying that Λ' cannot be a wind wheel algebra. This, along with [Mo, Theorem 6.6], implies that (Q', I') contains a node and, which is the desired result. □

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