Research Statement

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Introduction and Overview

Outline of my research. My primary research focuses on the Representation Theory of Algebras and its rich interactions with Combinatorics and Algebraic Geometry. Motivated by both foundational results and contemporary developments in these fields (see Section 1), I have developed a long-term research program that contributes to the resolution of several open conjectures posed over the past 10 years (see Section 2). More specifically, my project has evolved mainly around some challenging problems in the context of τ-tilting theory, stability conditions, lattice theory, and polyhedral geometry. Over the years, I have broadened and deepened this program, leading to substantial results with a lot of potential for further developments. This line of research forms the foundation of my work and defines a major trajectory for my future studies.

Beyond my central research agenda, I actively pursue collaborative projects that explore applications of Representation Theory in related disciplines. Some of my earlier projects of this nature are briefly mentioned below. Most recently, I have begun new interdisciplinary research, especially in the emerging connections with Topological Data Analysis, which I see as a promising direction for fruitful collaborative projects.

Publicity and impact of my research. According to Google Scholar, as of September 2025, my research articles, consisting of 13 manuscripts, have been cited over 240 times (more than 220 citations since 2020), reflecting the interest from the broader mathematical community in my topics of study. In particular, the long-term project I have developed around the *brick-Brauer-Thrall (bBT) conjectures* has proven a fertile ground for original research and has attracted attention from other researchers. Although still in its early stages, most recent citations of my work pertain to our studies on these problems. Moreover, a recent report ([AIR2, Section 3.2]) on developments in τ-tilting theory by its founders recognizes this line of research as a notable advance . I have also contributed to the publicity of these problems in several international research schools, workshops, and conferences, both as an organizer and a speaker, as detailed in my CV.

Content of this document. The primary focus of this document is my long-term research agenda that originated from my doctoral studies and developed significantly over the past 5 years. For the sake of clarity and brevity, some technical aspects of my work and the earlier studies are not discussed in details.

(1) **Section 1** provides an overview of my earlier work and **some original motivations in context**. In particular, I briefly recall a mutation phenomenon and a modern notion of finiteness that served as central motivations for the earlier objectives [Mo1, Mo2, MP1] of my long-term program. This also relates to some of my other projects [AI+, BD+, B-M+, KMS] in different directions.

(2) **Section 2** constitutes the core of this text and describes **the main objectives of my research**. This part is centered on several open conjectures, including those I posed during my Ph.D. studies and have pursued ever since, the so-called *brick-Brauer-Thrall (bBT) conjectures*. Here, I also highlight some future goals based on the major steps in [Mo3, MP2, MP3, MP4].

(3) **Section 3** briefly explains certain capacities of **my studies via interdisciplinary collaboration**. In this short section, I outline some problems that we study in my new ongoing collaborations in relation to persistence theory and their invariants within the framework of topological data analysis.

Setting & Notations. Throughout, k is assumed to be an algebraically closed field and A denotes a finite dimensional associative k-algebra with multiplicative identity. Without loss of generality, A is assumed to be basic and connected. Thus, by a theorem of Gabriel, we can always assume $A = kQ/I$, where Q is a finite connected quiver, and I is an admissible ideal in the path algebra kQ. Let $\operatorname{mod} A$ denote the category of finitely generated left A-modules. Unless specified otherwise, modules are considered up to isomorphism. For M in $\operatorname{mod} A$, let $|M|$ denote the number of non-isomorphic indecomposable summands of M in its Krull-Schmidt decomposition. We always assume $|A| = n$; that is, $\operatorname{mod} A$ has exactly n simple modules. For $\underline{d} \in \mathbb{Z}^n_{\geq 0}$, by $\operatorname{mod}(A, \underline{d})$ denote the representation variety consisting of all $M \in \operatorname{mod} A$ whose dimension vector is $\underline{d}$, where the general linear group $\operatorname{GL}(\underline{d})$ acts on $\operatorname{mod}(A, \underline{d})$ via conjugation. For X in $\operatorname{mod}(A, \underline{d})$, let O_X denote the $\operatorname{GL}(\underline{d})$-orbit of X, which is in bijection with the isoclass of X in $\operatorname{mod} A$. By $\overline{O}_X$ we denote the orbit closure of O_X, with respect to Zariski topology. We let $\operatorname{Irr}(A, \underline{d})$ denote the set of all irreducible components of $\operatorname{mod}(A, \underline{d})$, and set $\operatorname{Irr}(A) := \bigcup_{\underline{d} \in \mathbb{Z}^n_{\geq 0}} \operatorname{Irr}(A, \underline{d})$. For a given set, we say a property holds for *almost all* elements if it holds for all but at most finitely many elements of that set. For all the undefined terminology and unexplained details, I refer to our recent survey [MP5] and the references therein.

1. Earlier studies & Origins of my long-term project

1.1. **Mutation in (τ-)tilting theory.** In the past few decades, various mutation phenomena in mathematics and physics, and the conceptual connections between them, have motivated substantial fundamental research. Roughly speaking, one begins with a set of compatible objects satisfying certain desired properties and then iteratively replaces each object with a new one so that the resulting set remains compatible and preserves similar properties. This phenomenon provides the conceptual foundation for some aspects of my work, as it arises in tilting theory [AHK], cluster algebras [FZ1], and τ-tilting theory [AIR1].

For an algebra A of rank n, let $\operatorname{tilt}(A)$ denote the set of all basic tilting modules in $\operatorname{mod} A$; that is, $X \in \operatorname{tilt}(A)$ if and only if X is a rigid module of projective dimension ≤ 1, and $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$, where X_i's are non-isomorphic indecomposables. In particular, $|X| = n$ and $\operatorname{Ext}^1_A(X_i, X_i) = 0$, for $1 \leq i, j \leq n$. Then, ***mutation*** of X at an indecomposable summand, say X_1, is obtained by replacing that summand with a non-isomorphic indecomposable, say X_1', so that $Y = X_1' \oplus X_2 \oplus \cdots \oplus X_n$ also belongs to $\operatorname{tilt}(A)$. An important result (known as Bongartz's completion) implies that if X_1' exists, it must be unique. This process is captured in the mutation graph of tilting modules, denoted by $\mathtt{H}(\operatorname{tilt}(A))$, where the vertices correspond to elements of $\operatorname{tilt}(A)$, and each edge indicates a single mutation. Often seen as a deficiency of the mutation phenomenon in classical tilting theory, it is notable that $\mathtt{H}(\operatorname{tilt}(A))$ is not necessarily n-regular.

In their seminal work in 2014, Adachi-Iyama-Reiten [AIR1] introduced *τ-tilting theory*; a far-reaching generalization of the classical tilting theory that unifies and greatly extends some deep ideas from a wide range of topics (see [AIR2], and references therein). Here, τ denotes the Auslander-Reiten translation in $\operatorname{mod} A$. A module M is called *τ-rigid* if $\operatorname{Hom}_A(M, \tau M) = 0$, and it is ***$\tau$-tilting*** if additionally $|M| = n$. More generally, M is *support τ-tilting* if there exists an idempotent e in A such that M is a τ-tilting module over the quotient algebra $A/\langle e \rangle$. Let $\operatorname{s\tau-tilt}(\mathrm{A})$ denote the set of all basic support τ-tilting modules. Then, the Auslander-Reiten formula implies $\operatorname{tilt}(A) \subseteq \tau\text{-tilt}(A) \subseteq \operatorname{s\tau-tilt}(\mathrm{A})$. Among many other fundamental results in [AIR1], the authors generalized the mutation phenomenon from tilting modules to that of support τ-tilting modules, and considered the new mutation graph $\mathtt{H}(\operatorname{s\tau-tilt}(\mathrm{A}))$. In fact, they elegantly fixed the above-mentioned deficiency in the classical tilting theory and showed that $\mathtt{H}(\operatorname{s\tau-tilt}(\mathrm{A}))$ is always n-regular.

1.2. **τ-tilting finite algebras.** A central question in (τ-)tilting theory is whether the mutation process terminates after finitely many steps. In this context, an algebra A is called ***(τ-)tilting finite*** if it admits only finitely many isoclasses of basic (τ-)tilting modules; this occurs precisely when the corresponding mutation graph is finite. Meanwhile, recall that an algebra A is said to be *representation-finite* (rep-finite) if $\operatorname{ind}(A)$ is finite, where $\operatorname{ind}(A)$ denotes the set of (isoclasses of) all indecomposable modules in $\operatorname{mod} A$. Although rep-finite algebras are obviously τ-tilting finite, it is worth noting that τ-tilting finiteness does not imply rep-finiteness. Moreover, tilting finite does not imply τ-tilting finite.

Since τ-tilting finite algebras have finite and regular mutation graphs, they can be regarded as analogues of those cluster algebras with only finitely many clusters, already classified in [FZ2]. This analogy and the novelty of τ-tilting finiteness (in comparison with rep-finiteness and tilting finiteness) soon inspired extensive research in various directions. In fact, τ-tilting finite algebras exhibit rich combinatorial, homological, geometric, and lattice-theoretical structures (see [AIR1, AIR2, As1, As2, AI+, BD+, BKT, BST, DI+, EJR, GMM, FG, GLS, MP3, MSt, PPP]), and their characterizations have received great attention ([AMV, ALS, De, DIJ, MSt, Mi, Pl, STV, Wa, Zi], and references therein). Although it is highly desirable to determine which algebras are τ-tilting finite, a concrete classification of all τ-tilting finite algebras is out of reach. This naturally raised two important questions, which motivated my dissertation [Mo1]:

Q1: For which families of algebras A does A being τ-tilting finite implies that A is representation-finite?
Q2: How can one systematically determine if a representation-infinite algebra is in fact τ-tilting (in)finite?

In [Mo1] I used some important results on τ-tilting theory and reduction arguments to systematically treat these questions. Recall that an algebra A is *minimal representation-infinite* (min-rep-inf, for short) if A is rep-infinite but all proper quotients of A are rep-finite. These algebras are historically important and studied for several decades (see [Bo] and references therein). In relation to the above questions, note that:

(1) Min-rep-inf algebras are the first family for which the problem of τ-tilting finiteness is nontrivial.
(2) Each representation-infinite algebra A has at least one quotient algebra (and possibly several) which is min-rep-inf. Thus, if one of the min-rep-inf quotients of A is τ-tilting infinite, then so is A.

Adopting a wide range of tools, and making substantial use of some results of [Bo, DIJ, Ri], in [Mo1] I fully settled the problem of τ-tilting (in)finiteness for a large family of min-rep-inf algebras. In fact, for the remaining min-rep-inf algebras, there is no hope for full and concrete results similar to what I obtained.

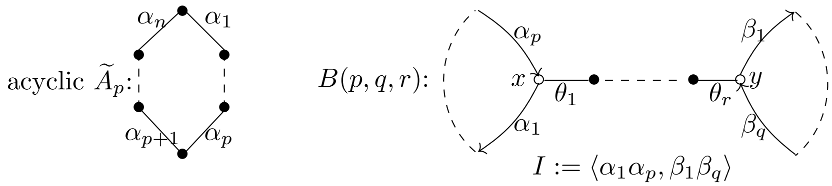

Figure 1. Full classification of min-rep-inf biserial algebras which are τ-tilting infinite [Mo2].

Theorem 1.1 ([Mo1, Mo2, Mo3]) *Let $A = kQ/I$ be a minimal representation-infinite algebra. Then,*

(1) If A is biserial, then A is τ-tilting infinite if and only if (Q, I) is one of the bound quivers in Figure 1, where $n, p, q, r \in \mathbb{Z}_{\geq 1}$, and $\widetilde{A}_p$ is acyclic (Explicit bound quivers are given in [Mo2]*).*

(2) If $A = kQ/I$ is non-distributive, then A is τ-tilting infinite if and only if Q has a sink or a source vertex (Explicit bound quivers are given in [Mo3]*).*

Building on the above ideas, in [Mo2, Mo3] I also strengthened and extended some earlier results on gentle and non-distributive algebras, previously treated independently in [Pl] and [CKW].

1.3. **Minimal τ-tilting infinite algebras.** Inspired by my reductive approach, in my Ph.D. thesis I also introduced a modern analogue of some classical notion. Algebra A is said to be **minimal τ-tilting infinite** algebra if it is τ-tilting infinite but each proper quotient of A is τ-tilting finite. This leads to a novel class of algebras exhibiting rich properties, and inspired the related concept of *minimal tilting-infinite algebras*, defined analogously. Observe that, unlike representation-finiteness and τ-tilting finiteness, tilting finiteness is not always preserved under taking quotients. This underscores the depth and subtlety of the structures involved, and indicates that the following theorem is highly nontrivial (for details, see [Mo3, MP1]).

Theorem 1.2 ([Mo3, MP1]) *An algebra A is minimal τ-tilting infinite if and only if it is minimal tilting infinite. Moreover, for each minimal τ-tilting infinite algebra A of rank n, we have*

(1) A is central and admits no projective-injective module, and its bound quiver has no nodes.
(2) For almost all τ-rigid A-modules, the projective dimension is exactly one.
(3) The mutation graph $\mathtt{H}(\mathrm{tilt}(A))$ is infinite and n-regular at almost all vertices.

We also gave a full and combinatorial classification of biserial minimal τ-tilting infinite algebras [MP2], that can be seen as the modern analogue of the algebras belonging to $\mathrm{Mri}(\mathfrak{F}_{\mathrm{B}})$, treated in [Ri].

Theorem 1.3 ([Mo2, MP2]) *A biserial algebra $A = kQ/I$ is minimal τ-tilting infinite if and only if (Q, I) is a generalized barbell algebra, or else $A = k\widetilde{A}_m$, for an acyclic quiver $\widetilde{A}_m$ of affine type A. In particular, every biserial minimal τ-tilting infinite algebra is a gentle algebra (Explicit bound quivers are given in* [MP2]*).*

2. Main objectives, Open conjectures & Future goals

This section presents my extensive work on a series of important conjectures proposed between 2015-2024, nowadays referred to as the *brick-Brauer-Thrall (bBT) conjectures.* I first recall my original conjectures from [Mo1], and then discuss a broader range of open problems central to my ongoing and future research. For clarity, most statements are formulated in terms of bricks, with references included for more technical details.

Recall that a (not necessarily finitely generated) left A-module X is a **brick** if $\mathrm{End}_A(X)$ is a division algebra; that is, each nonzero A-endomorphism $f : X \to X$ is invertible. Let $\mathrm{brick}(A)$ denote the set of all isoclasses of bricks in $\mathrm{mod}\, A$. Note that $X \in \mathrm{brick}(A)$ if and only if $\mathrm{End}_A(X) \simeq k$; such modules are also known as *Schur representations.* An algebra A is called *brick-finite* if $\mathrm{brick}(A)$ is a finite set. Since $\mathrm{brick}(A) \subseteq \mathrm{ind}(A)$, representation-finite algebras are brick-finite; however, the converse does not hold in general. Bricks and their structural properties play an important role in various areas of current research. For instance, an important result of [DIJ], known as "brick-τ-rigid correspondence", implies A is τ-tilting finite if and only if A is brick-finite. Some other applications of bricks in stability conditions, wide subcategories, g-vector fans, generic modules, etc. are considered in my research; see [MP5] and references therein.

2.1. **My brick-analogue of Second Brauer-Thrall Conjecture.** The celebrated Brauer-Thrall (BT) conjectures posed in late 1950's inspired many directions of research in representation theory; including covering theory, tilting theory, Auslander-Reiten theory, degeneration theory, boces, generic modules, etc. The BT conjectures were mainly concerned with the distribution of indecomposables, including the following challenging problem that was open for over 30 years (For historical details see [Bo] and references therein).

- *Second Brauer-Thrall Conjecture (2nd BT)* [Brauer & Thrall, 1950's]: If A is rep-infinite, there exists an infinite sequence $d_1 < d_2 < \cdots$ of integers with an infinite family of non-isomorphic indecomposables of dimension d_i, for each d_i.

Inspired by this problem and my earlier studies of τ-tilting theory and geometry of representation varieties, in the final stage of my Ph.D. program, I posed two conjectures which are still open in full generality. Properly understood, my stronger conjecture is a modern analogue of the 2nd BT conjecture; therefore, nowadays it is called the "*Second brick-Brauer-Thrall (2nd bBT) conjecture*". I emphasize that a verbatim brick-analogue of 2nd BT conjecture is not true (for details, original articulation, and motivations, see [Mo1, Chapter 6]).

- ***brick-analogue of 2nd BT Conjecture (2nd bBT)*** **[M, 2019]**: If A is brick-infinite, there exists (at least) one integer d with an infinite family of non-isomorphic bricks of dimension d.

In my thesis and subsequent work, I verified the above conjecture for various families of algebras. Later, this conjecture also appeared in [STV]. Some geometric observations imply that an affirmative answer to 2nd bBT also settles the following open problem. Before stating that, note if every $X \in \text{brick}(A)$ is τ-rigid, then A is necessarily rep-finite (see [MP3]). For details, see [Mo1, Conjecture 6.0.1].

- ***Rigid-bricks Conjecture*** **[M, 2019]**: If every $X \in \text{brick}(A)$ is rigid, then A is τ-tilting finite.

Some important reductions in the above conjectures are achieved in my recent work. To state some of them, first recall that an algebra A is said to be ***minimal brick-infinite*** (min-brick-inf, for short) if it is brick-infinite but all proper quotients of A are brick-finite. These algebras coincide with the minimal τ-tilting infinite algebras (see Theorem 1.2). Observe that each brick-infinite min-rep-inf algebra is min-brick-inf, but the converse is not true in general, and some standard arguments imply that the min-brick-inf algebra play a decisive role in the above conjectures. Some of our earlier results on these algebras are summarized below.

Theorem 2.1 *The 2nd bBT Conjecture (hence, the Rigid-brick Conjecture) holds for the following families:*

(1) All non-distributive minimal representation-infinite algebras (For a more refined version, see [Mo3]*).*
(2) All minimal brick-infinite algebras A with infinitely many unfaithful bricks (For details, see [MP1]*).*
(3) All biserial algebras (For more refined versions, see [MP2]*).*

The above results provided strong evidence for my open conjectures and, as summarized in the following statement, we obtained an important reduction in their studies. This reduction was particularly decisive in our follow-up work, in which we treated a wider range of open problems.

Corollary 2.2 *To prove the* 2nd *bBT Conjecture (and Rigid-bricks Conjecture), it suffices to verify it for the minimal brick-infinite algebras over which almost all bricks are faithful (i.e., they have trivial annihilator).*

2.2. **Open conjectures and future goals.** As discussed in [AIR2, MP5], the 2nd bBT Conjecture relates to a range of classical and modern research areas, where it has many profound applications. In [MP4] we introduce a unified treatment of several fundamental conjectures posed in recent years [CKW, De, En, Mo1, MP2, STV, Pf]. Thanks to the conceptual linkages between these problems and the central role of bricks, nowadays we generally refer to them as *brick-Brauer-Thrall (bBT) Conjectures*. Observe that most of the problems listed below were originally formulated and treated in more technical frameworks. However, for the sake of clarity, here I present (an equivalent version of) them in terms of bricks and their properties. The following conjectures are listed in order of their appearance in the literature (based on arXiv appearance). For all undefined terminology, notation, and technical details, see [MP5] and references therein.

bBT Conjectures (*brick-Brauer-Thrall Conjecture*): With the same setting as before,

(I) ***g-vectors Conj.*** **(2017)**: A is brick-infinite $\Leftrightarrow$ there is a rational ray outside the τ-tilting fan of A.
(II) ***2nd bBT Conj.*** **(2019)**: A is brick-infinite $\Leftrightarrow$ a brick component $\mathcal{Z} \in \text{Irr}(A)$ has no open orbit.
(III) ***Semibrick Conj.*** **(2021)**: A is brick-infinite $\Leftrightarrow$ A admits an infintie semibrick.
(IV) ***Generic-brick Conj.*** **(2022)**: A is brick-infinite $\Leftrightarrow$ A admits a generic brick.
(V) ***Stability Conj.*** **(2023)**: A is brick-infinite $\Leftrightarrow$ for some $\theta \in K_0(\text{proj}\, A)$, there are infinitely many θ-stable modules of the same dimension.

These conjectures indicate the importance of bricks and their structural properties. It is notable that, (I) first appeared as a question in [De], (II) was first posed in [Mo1, Mo3] and later in [STV], (III) was first conjectured in [En], and (IV) was originally formulated in [MP2] and later treated in [BPS] in the study of infinite-dimensional bricks. Furthermore, (V) is a combination of two conjectures that appeared, respectively, in [CKW] and [Mo3]. In fact, the conjecture posed in [CKW] first appeared in 2013 and predates the above bBT Conjectures (for details, see [Mo1, Chapter 6]). This version of (V) formulated above also appeared in [Pf]. For a detailed treatment of these problems and more historical remarks, see [MP4, Section 2].

Before listing some future objectives, I remark that our results in [MP4] imply that bBT Conjectures (I), (II), (III), and (V) hold for any algebra with a generalized standard component (for the importance of such algebras, see [MP4, MSk]). Consequently, all bBT Conjectures are true for any algebra with a preprojective or a preinjective component. The next theorem indicates how our systematic treatment of these problems has resulted in unification and reductions, which suggests a promising approach to our future studies.

Theorem 2.3 *With the same setting and notation as before, we have the following:*

[MP2] *All of the bBT Conjectures hold for biserial algebras.*
If A is a biserial algebra of rank n, all of bBT Conjectures hold if and only if A admits an infinite family of bricks of dimension less than $2n+1$ (For details and related studies, see [MP2, MP4, STV]*).*

[MP4] *All of the bBT Conjectures are equivalent for tame algebras.*
To verify any of the bBT Conjectures for tame algebras, it suffices to prove it for minimal brick-infinite tame algebras A with almost all bricks faithful. That being the case, bBT Conjectures hold if and only if A admits a homogeneous brick (For details and related studies, see [BPS, MP4, MP5, Pf]*).*

Another key insight is that many of the bBT Conjectures can be approached through the framework of Hom-orthogonal modules (see [MP4]), providing a significantly more tractable and conceptual perspective. Through this, we characterized τ-tilting finite algebras as conceptual generalizations of local algebras, and proved that the Second bBT Conjecture implies the Semibrick Conjecture. This line of reasoning led to the formulation of the following elementary conjecture. Recall that a family $\{X_i\}_{i\in I}$ in $\operatorname{mod} A$ is said to be *pairwise Hom-orthogonal* if $\operatorname{Hom}_A(X_i, X_j) = \operatorname{Hom}_A(X_j, X_i) = 0$, for all distinct $i, j \in I$.

- ***Hom-orthogonal Conjecture (2024)***: For any algebra A of rank n, the following are equivalent:
  (1) $\operatorname{mod} A$ contains an infinite family of pairwise Hom-orthogonal modules of the same dimension.
  (2) $\operatorname{mod} A$ contains $n+1$ pairwise Hom-orthogonal bricks in $\operatorname{mod} A$.
  (3) A is brick-infinite.

Some recent results imply the implications (1) $\rightarrow$ (2) $\rightarrow$ (3), but the reverse implications are open in general (see [MP4]). Notably, over tame algebras, this is equivalent to all of the above bBT Conjectures.

The elementary formulation of the bBT Conjectures achieved in our recent work, together with the new results, highlights the breadth, depth, and significance of these problems across multiple areas of research. These conjectures and their applications have also been developed within the geometric framework of representation varieties, with important links to recent studies on τ-regular components [BS, MP4, Pf]. Moreover, they can contribute significantly to the study of several modern notions of tameness, including the brick-tame, g-tame, and E-tame algebras [AI, BD, MP5, PY]. Our unified approach to the bBT Conjectures, along with our techniques (such as τ-convergence in [MP3] and brick-splitting torsion theory in [AI+]), has led to a rich collection of fundamental tools and problems that I plan to explore in my future work. Several concrete intermediate steps in this long term research program are laid out in [AI+, MP3, MP4, MP5].

3. Some interdisciplinary research

As a more recent direction of my research, I have developed interest in the interdisciplinary collaborative projects between Representation Theory (RT) and Topological Data Analysis (TDA). As briefly outlined below, I have started some joint projects that examine the interplay between these topics, with a particular focus on Multiparameter Persistence Theory (MPT), the mathematical foundation of TDA. Central to this work are persistence modules, which are typically viewed as functors from partially ordered sets to the category of vector spaces. As a result, many challenging problems in MPT can be treated through the representation of bound quivers, and a range of tools from representation theory are proving to be highly effective in analyzing key phenomena in TDA (for further details, see [Ou] and related references).

Roughly speaking, one-parameter persistence modules can be completely classified using the representation theory of type A quivers. As a result, their structure and stability properties are well-understood. In contrast, for $n > 1$, multiparameter persistence modules quickly enter the realm of "wild" representation type. This makes a systematic study of multiparameter persistence theory particularly challenging, as many of the tools and results from the one-parameter setting do not extend to higher dimensions (see [CZ]). In particular, in the absence of a full description of all indecomposables, for $n > 1$, current research focuses on developing and understanding invariants of persistence modules. In this context, recent interactions between RT and TDA have garnered significant attention, leading to notable advances in both fields (for details, see [BL] and references therein). While restricted to certain settings, one can further employ a wider range of tools in the study of MPT. Notably, this includes the combinatorial machinery of gentle algebras [BO+], as well as emerging techniques from geometric representation theory [FJ+].

Inspired by my recent discussions with some experts in TDA and related topics, I have identified new ways in which my extensive research on bricks and their properties can be applied and further developed to study some problems in the context of MPT. More specifically, in an ongoing collaborative project, we explore a novel concept of directedness in relation to certain persistence modules. In particular, by focusing on the key characteristics of *spread modules* (also known as interval modules), we introduce and investigate the notion of *spread-directed posets*.

To better frame our main problem, it is important to note that spread modules form a specialized subset of bricks. Hence, the notion of spread-directed poset is a generalization of the brick-directed algebras recently studied in [AI+], which, in turn, extended the classical notion of representation-directed algebras. Furthermore, spread modules play a pivotal role in the algebraic and homological treatments of MPT (see [AE+, BBH], and related references). In light of this, our current interdisciplinary collaboration aims to address the following question: *Which posets (potentially infinite) are spread-directed? In other words, which posets do not contain cycles of nonzero, non-invertible maps between spread modules?*

Building on previous work studying spread modules over ladder posets (e.g., [EY]), we have already obtained some promising results for these posets. In fact, our findings suggest that spread-directed posets provide a novel generalization of brick-directed and representation-directed posets. Moreover, we anticipate that the study of challenging problems in MPT will become significantly more manageable when restricted to spread-directed posets. Ultimately, we expect that our results will offer valuable new insights into both the theoretical and computational aspects of TDA.

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